ENGINEERING TRIPOS PART IIB

Tuesday 1 May 2012 2.30 to 4

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 Adaptive filters are commonly used for prediction. The aim is to form a linear predictor of the real valued signal $\{x(n)\}$ from noisy measurements:

$$u(n) = x(n) + v(n)$$

where $v(n)$ is zero-mean white noise that is uncorrelated with $x(n)$. The variance of $v(n)$ is σ_v^2.

(a) Describe in detail the Least Mean Square (LMS) algorithm to implement an M-step predictor for $x(n)$. [20%]

(b) The block LMS algorithm is defined by:

$$\mathbf{h}(n+L) = \mathbf{h}(n) + \mu \frac{1}{L} \sum_{l=0}^{L-1} \left(d(n+l) - \mathbf{h}(n)^{\mathrm{T}} \mathbf{u}(n+l) \right) \mathbf{u}(n+l)$$

where this corresponds to the standard LMS algorithm when the block size $L = 1$. In the block version, the update of the column vector of filter coefficients $\mathbf{h}(n)$ is done after accumulating L samples of the desired response $d(n+l)$ and the filter output $\mathbf{h}(n)^{\mathrm{T}}\mathbf{u}(n+l)$.

(i) By evaluating the behaviour of $E(\mathbf{h}(n))$ determine the conditions on the step-size μ for the block LMS to converge in mean. Why might it be useful to have $L > 1$ as opposed to $L = 1$? [40%]

(ii) Characterise the limit of $E(\mathbf{h}(n))$ of the block LMS when applied to the prediction problem in part (a). [40%]

2 We have repeated observations of a random variable x through

$$y(n) = x + v(n) \quad \text{for } n = 1, 2, \ldots$$

where $\{v(n)\}$ is an independent and identically distributed zero-mean scalar noise sequence, independent of x, with variance $E\left(v(n)^2\right) = \sigma_v^2$. Also, $E(x) = 0$ and $E\left(x^2\right) = \sigma_0^2$. The aim is to estimate x using the observations.

(a) Find the mean square error (MSE) of the sample mean estimate,

$$\frac{1}{n}\sum_{i=1}^{n} y(i)$$

[20%]

(b) Let $\widehat{x}(n-1)$ be the sample mean estimate of x using $\{y(1), \ldots, y(n-1)\}$. Show that the sample mean estimate at time n can be written as

$$\widehat{x}(n) = K(n)\ (y(n) - \widehat{x}(n-1)) + \widehat{x}(n-1) \tag{1}$$

and state the value of $K(n)$. [5%]

(c) Consider a general estimator $\widehat{x}(n)$ of x satisfying equation (1).

(i) Let $\sigma(n)^2 = E\left((\widehat{x}(n) - x)^2\right)$ and find the recursion that is satisfied by $\sigma(n)^2$. [50%]

(ii) Find the value of the gain $K(n)$ at time n that minimises $E\left((\widehat{x}(n) - x)^2\right)$. [25%]

3 The P-th order Autoregressive (AR) model, or AR(P) model, of a discrete-time zero-mean stationary random signal is defined by:

$$x_n = -\sum_{i=1}^{P} a_i x_{n-i} + \sigma w_n$$

where $\{w_n\}$ is a zero-mean white noise process with variance 1.

(a) Discuss how this model may be used to represent the power spectrum of a general random process $\{x_n\}$. [15%]

(b) Show that the autocorrelation sequence $R_{XX}[k]$ of $\{x_n\}$ also satisfies a similar difference equation. [40%]

(c) Show how the autocorrelation sequence can be used to estimate the parameters of an AR(P) model and when $P = 2$, these equations simplify to

$$\begin{aligned} \begin{bmatrix} R_{XX}[0] & R_{XX}[-1] \\ R_{XX}[1] & R_{XX}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= -\begin{bmatrix} R_{XX}[1] \\ R_{XX}[2] \end{bmatrix}, \\ R_{XX}[0] + a_1 R_{XX}[1] + a_2 R_{XX}[2] &= \sigma^2 \end{aligned}$$

[20%]

(d) The autocorrelation values of a particular signal are estimated to be

$$\widehat{R}_{XX}[0] = 4.8, \qquad \widehat{R}_{XX}[1] = -1.2, \qquad \widehat{R}_{XX}[2] = 1.8$$

Determine the estimates of the AR model parameters a_i and noise variance σ^2. [15%]

(e) Sketch the estimated power spectrum of the signal. [10%]

4 We would like to estimate the power spectrum of the AR(2) process

$$x_n + a_1 x_{n-1} + a_2 x_{n-2} = \sigma w_n$$

where $\{w_n\}$ is a zero-mean white noise process with variance 1. Measurements of x_n are noisy and what is actually observed is

$$y_n = x_n + v_n$$

where $\{v_n\}$ is a Moving Average (MA) process

$$v_n = b_0 e_n + b_1 e_{n-1}$$

where $\{e_n\}$ is a zero-mean white noise process with variance 1. Sequences $\{w_n\}$ and $\{e_n\}$ are independent.

(a) Find the relationship between the autocorrelation functions of the sequences $\{x_n\}$, $\{v_n\}$ and $\{y_n\}$. [20%]

(b) Write down the power spectrum of $\{x_n\}$, $\{v_n\}$ and $\{y_n\}$. [25%]

(c) Based on measurements of $\{v_n\}$, the power spectrum of $\{v_n\}$ is estimated to be

$$\widehat{S}_V(e^{j\omega}) = 2 + 2\cos\omega$$

Estimate the coefficients b_0 and b_1 and then calculate the autocorrelation function of $\{v_n\}$.

[35%]

(d) From measurements of y_n, you are given the following estimates of its autocorrelation function:

$$\widehat{R}_{YY}[0] = 4.74, \qquad \widehat{R}_{YY}[1] = 0.54, \qquad \widehat{R}_{YY}[2] = 1.41$$

Estimate the power spectrum of $\{x_n\}$. You may use the following Yule-Walker equations

$$\begin{bmatrix} R_{XX}[0] & R_{XX}[-1] \\ R_{XX}[1] & R_{XX}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} R_{XX}[1] \\ R_{XX}[2] \end{bmatrix},$$

$$R_{XX}[0] + a_1 R_{XX}[1] + a_2 R_{XX}[2] = \sigma^2$$

[20%]

END OF PAPER