ENGINEERING TRIPOS PART IIA
ENGINEERING TRIPOS PART IIB

Wednesday 2 May 2012 2.30 to 4

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: 4M12 Data Sheet (3 sides).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

gnw04

1 The strong form of the Stokes equations reads

$$-\nabla \cdot 2\nu\nabla^s \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad (1)$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad (2)$$

where $\boldsymbol{u}$ is the velocity, $\nu > 0$ is the viscosity, p is the pressure, $\boldsymbol{f}$ is a prescribed forcing term and $\nabla^s \boldsymbol{u} = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2$.

(a) Using index notation, derive the Stokes equations by considering mass and linear momentum balances for a body V. Recall that the stress is given by $\boldsymbol{\sigma} = 2\nu\nabla^s \boldsymbol{u} - p\boldsymbol{I}$ and traction $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$, where $\boldsymbol{n}$ is the outward unit normal vector to a surface. [30%]

(b) Equation (1) is sometimes written as $-\nu\nabla \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{f}$. For an incompressible flow, show that the two expressions are equivalent when ν is constant. [30%]

(c) Using Lagrange multipliers, show that minimising

$$J = \int_V \nu\nabla^s \boldsymbol{u} : \nabla^s \boldsymbol{u} - \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}V$$

subject to $\nabla \cdot \boldsymbol{u}$ everywhere in V is equivalent to solving the strong form of the Stokes problem. Use this to provide an interpretation for the pressure in incompressible flows.

Hint: $\boldsymbol{A} : \boldsymbol{B} = A_{ij}B_{ij}$.

[40%]

2 You are provided with an inextensible string of length $\pi a/2$ which is to be hung at its ends from two anchors that are at the same height. The density $\rho(s)$ of the string varies along its length such that the hanging string forms a circular arc of radius a. The distance along the string s is measured from its centre.

(a) What must the distance between the two anchors be? [20%]

(b) Provide expressions for the potential energy of the system and any constraints that must be satisfied in terms of integrals with respect to ds. Make s the independent variable and use $y = y(s)$. [30%]

(c) Via minimisation of potential energy, show that the necessary $\rho(s)$ for the string to hang in a circular arc is proportional to $\sec^2(s/a)$. [50%]

3 A domain is given by a half-plane, with a semi-circle of radius a centred at the origin removed (see Fig. 1). Consider the Laplace equation on this domain

$$\begin{cases} \nabla^2\phi = 0 & \text{for } r > a, 0 < \theta < \pi \\ \partial\phi/\partial n = 0 & \text{on the wall and semi-circle} \\ \phi - (x^2 - y^2) = O(1) \text{ at most} & \text{as } r \to \infty \end{cases}$$

where (x, y) are the Cartesian coordinates and (r, θ) the polar coordinates.

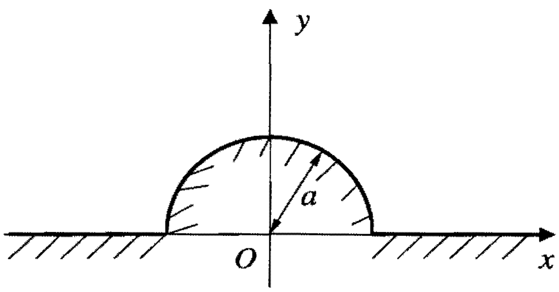

Fig. 1

(a) Assuming a solution of the form

$$\phi = R(r)H(\theta)$$

use the method of separation of variables to deduce differential equations for θ and for R, with a separation constant. [20%]

(b) Find the boundary conditions for R and θ. [20%]

(c) Show that solutions for H are of the form

$$\text{constant} \times \cos(n\theta)$$

where n is a positive integer. [20%]

(d) Find the solutions of R for each n in (c). [20%]

(e) Determine the unique solution for ϕ. [20%]

4 (a) Consider a 2×2 system of quasi-linear, first-order partial differential equations with independent variables t and x

$$\frac{\partial u}{\partial t} + a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial v}{\partial x} = 0$$

where a coefficient $a_{ij} = a_{ij}(u,v)$. State the condition for which this system is hyperbolic and explain how the concept of Riemann invariants can be used to solve this hyperbolic system. [30%]

(b) Consider the system

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ m \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} m \\ \frac{m^2}{\rho} + \rho a^2 \end{pmatrix} = 0 \qquad (3)$$

where a is a positive constant. Show that the system is hyperbolic and find the eigenvalues of its characteristic equation and corresponding left eigenvectors. [30%]

(c) Find ρ and m in Eq. (3) in terms of the Riemann invariants. [40%]

END OF PAPER