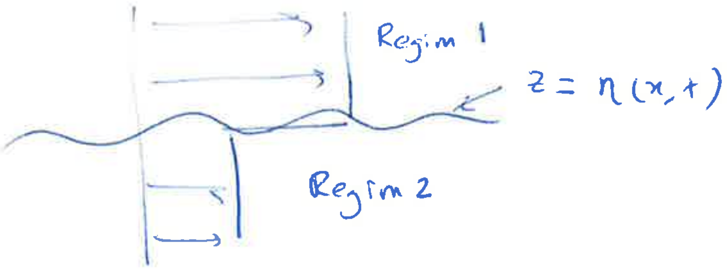
1. (a)



Region 1:

$$\phi_1 = U_1 x + f(z) e^{ikx} e^{st} \qquad (1)$$

Region 2:

$$\phi_2 = U_2 x + g(z) e^{ikx} e^{st} \qquad -(2)$$

ϕ satisfies the potential flow equation: $\nabla^2 \phi = 0$.

$$\therefore \quad \phi_1 = U_1 x + \left(A e^{-kz} + C e^{ky}\right) e^{st} e^{ikx} \qquad -(3)$$

for the solution to be bounded.

$$\phi_2 = U_2 x + \left(B e^{kz} + D e^{-ky}\right) e^{st} e^{ikx} \qquad -(4)$$

Boundary conditions

(I) η is continuous at the interface.

$$\frac{D\eta}{Dt} = \frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} \qquad -(5)$$

$$\Rightarrow \quad -Ak e^{-kz}\Big|_{z=\eta} = s\eta_0 + ikU_1 \eta_0 \qquad -(6)$$

$$Bk e^{kz}\Big|_{z=\eta} = s\eta_0 + ikU_2 \eta_0 \qquad -(7)$$

Linearising the boundary condition (BC) $\Rightarrow$ applying BC at $z = 0$.

$$\therefore \quad -Ak = n_0 (s + ik U_1)$$

$$Bk = n_0 (s + ik U_2)$$

$$\Rightarrow \quad -\frac{B}{A} = \frac{s + ik U_2}{s + ik U_1} \qquad - (8)$$

(II) Pressure boundary condition.

Neglecting gravity Bernoulli's equation is given by

$$p = p_0 - \rho \frac{\partial \phi}{\partial t} - \frac{1}{2} \rho |\underline{u}|^2$$

Form eq^n's (3) & (4); retaining only linear terms in perturbation:

$$p_1 = p_{0_1} - \rho s A e^{-kz} \underbrace{e^{st} e^{ikz}}_{E} - \frac{1}{2} \rho U_1^2 - \frac{1}{2} \rho \cdot 2 U_1 A i k e^{-kz} E$$

$$p_2 = p_{0_2} - \rho s B e^{kz} E - \frac{1}{2} \rho U_2^2 - \frac{1}{2} \rho\, 2 U_2 B i k e^{kz} E$$

BC is $p_1 = p_2$ at $z = 0$.

$$\Rightarrow \quad p_{0_1} - \frac{1}{2} \rho U_1^2 = p_{0_2} - \frac{1}{2} \rho U_2^2$$

$$\& \quad \rho s A E + \rho U_1 i k A E = \rho s B E + \rho U_2 i k B E$$

$$\Rightarrow \quad A(S + ikU_1) = B(S + ikU_2) \qquad -(9)$$

Using Eq (8):

$$(S + ikU_1) = -\frac{(S+ikU_2)}{(S+ikU_1)}(S+ikU_2)$$

$$(S+ikU_1)^2 = -(S+ikU_2)^2$$

$$\Rightarrow \quad S + ikU_1 = \pm i\,(S + ikU_2)$$

or $\quad S(1 \mp i) = -ikU_1 \mp kU_2$

$$S = \frac{-k(iU_1 \pm U_2)\,(1 \pm i)}{(1 \mp i)\,(1 \pm i)}$$

$$S = \frac{1}{2}\left\{ -ikU_1 \mp kU_2 \pm kU_1 - ikU_2 \right\}$$

$$\boxed{S = -\frac{1}{2}ik(U_1 + U_2) \pm \frac{1}{2}k(U_1 - U_2)}$$

1. (b) $S = S_r + iS_i$ $\xi = e^{St + ikx}$.

phase speed $= \dfrac{-S_i}{K}$

$$= \boxed{\frac{1}{2}(U_1 + U_2)}$$

(c) S has one root that is always positive.

Therefore this ~~system~~ flow is unstable for all wavenumbers.

(i) Boundary layer flow over a concave wall

(a) Rayleigh's criterion for inviscid, incompressible swirling flows shows that this flow is unstable. In other words Γ^2 decreases with radius r. A particle of fluid displaced normal to the surface has less velocity than the surrounding particles. The local pressure gradient provides the centripetal acceleration to keep the surrounding particles in equilibrium. However, because the displaced particle has a lower velocity, the pressure gradient exceeds the central acceleration required to keep the particle in equilibrium, so the particle continues to move away from the surface, leading to the instability.

(b) Viscosity has a stabilising influence.

(c) The instability depends on the momentum thickness θ, the radius of curvature R, kinematic viscosity ν and the free stream velocity U_∞. Therefore the relevant independent nondimensional groups are: $U_\infty\theta/\nu$ and θ/R. An appropriate number describing the instability for laminar flows is

$$\frac{U_\infty \theta}{\nu}\left(\frac{\theta}{R}\right)^{1/2}$$

(d) The flow breaks up into streamwise vortices known as Gortler vortices.

(ii) a shear flow with velocity profile $U(z) = U_0 \tanh(z/\delta)$

(a) This flow has an inflexion point ($U'' = 0$ at $z = 0$) and is inviscidly unstable by Rayleigh's inflexion point theorem.

(b) Viscosity has a stabilising influence.

(c) The relevant number describing the instability is the Reynolds number $\delta U_0/\nu$. The flow is unstable above a critical Reynolds number

(d) The flow develops Kelvin-Helmholtz instability in which the shear layer rolls up, the disturbance grows and eventually leads to turbulence.

(iii) a thin layer of liquid between two horizontal plates heated from below

(a) This instability is buoyancy driven. If a fluid particle is raised from the bottom plate, it has a lower density and is hence lighter than the surrounding fluid. Thus because of the buoyancy force it continues to rise upwards and is hence unstable to the perturbation.

(b) Viscous and heat diffusion are stabilising influences.

(c) The variables affecting the instability are: the temperature difference between the plates $T_0 - T_1$, separation between the plates d, coefficients of kinematic viscosity ν and thermal diffusion κ, acceleration due to gravity g and coefficient of thermal expansion α. The relevant nondimensional numbers are the Rayleigh number,

$$Ra = \frac{g\alpha(T_0 - T_1)d^3}{\nu\kappa}$$

(d) The flow breaks up into convection rolls beyond the critical value of the Rayleigh number. The rolls are steady. The axes of the rolls are perpendicular to the short sides of a rectangular container.

(iv) a thin layer of liquid with a free upper surface and heated from below

(a) In addition to the variables identified above, surface tension plays a key role in the stability. If a blob of fluid is raised to the surface then because it is hotter than the surrounding fluid, it would locally reduce the surface tension and because of surface traction, it would get pulled outwards, thus amplifying the initial motion. This is an additional mechanism of instability on top of the one discussed in (iii).

(b) Viscous and heat diffusion are stabilising influences

(c) The relevant nondimensional group is the Maragoni number

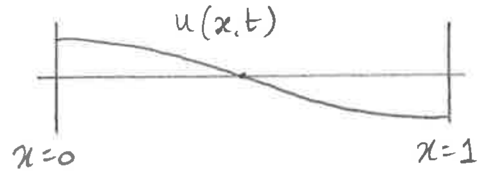
(d) The fluid breaks up into steady hexagonal cells, fluid rises through the centre and sinks at the edges of the cells.

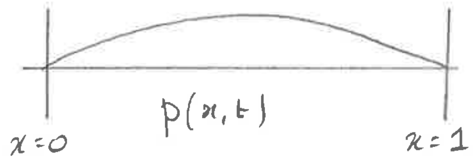
3

(a) boundary conditions: p and $\frac{\partial u}{\partial x}$ are zero at $x = 0$ and $x = 1$

if $u(x,t) = U \cos(\pi x)$ then $\frac{\partial u}{\partial x} = -\pi U \sin(\pi x)$, which $= 0$ at $x = 0$ and 1.

if $p(x,t) = P \sin(\pi x)$ then $p = 0$ at $x = 0$ and 1 by inspection.





The tube cannot support a longer wave, but it could support shorter waves.

(b) $q(t) = \alpha u(x_f,t) + \beta p(x_f,t)$; α and β unknown, both $\ll 1$

$$\left.\begin{aligned} \dot{U} + \pi P &= 0 \\ \dot{P} - \pi U &= aU + bP \end{aligned}\right\} \quad \text{where} \quad \begin{aligned} a &= 2\alpha \cos(\pi x_f)\sin(\pi x_f) \\ b &= 2\beta \sin(\pi x_f)\sin(\pi x_f) \end{aligned}$$

Assume solutions of the form $U = U_0 e^{st}$ and $P = P_0 e^{st}$

$\Rightarrow \dot{U} = sU$ and $\dot{P} = sP$

Substitute in to the governing equations:

$$(sU_0 + \pi P_0)e^{st} = 0$$
$$(sP_0 - \pi U_0)e^{st} = (aU_0 + bP_0)e^{st}$$

These can be written:

$$\begin{bmatrix} s & \pi \\ -\pi - a & s - b \end{bmatrix} \begin{bmatrix} U_0 \\ P_0 \end{bmatrix} = 0$$

For this to be satisfied for general non-trivial solutions $[U_0 \ P_0]^T$, the determinant of the matrix must be zero:

$$s(s-b) + \pi(\pi + a) = 0$$
$$\Rightarrow s^2 - bs + \pi(\pi + a) = 0$$
$$\Rightarrow s = \frac{b \pm \left(b^2 - 4\pi(\pi + a)\right)^{1/2}}{2}$$

where $a \ll 1$ and $b \ll 1$ so the argument of the square root term is -ve.

growth rate $=$ Real(s) $= b/2$

frequency $=$ Im(s) $= \left(\pi^2 + \pi a - \frac{b^2}{4}\right)^{1/2}$

(c) The heat release fluctuation at the wire is $q(t)$. If it is positive during moments of higher-than-average pressure and negative during moments of lower-than-average pressure then work is done on the air in the duct, increasing the acoustic waves' amplitude. (The work done is the area within the loop in figure 2.)

The heat release fluctuation is $q(t) = \alpha u(x_f, t) + \beta p(x_f, t)$. The component of q that is in phase with the pressure is $\beta p(x_f, t)$. The other component is in phase with the velocity which, by the first governing equation $\dot{u} + \pi P = 0$, is $90°$ out of phase with the pressure.

The growth rate is only affected by the component of $q(t)$ that is in phase with the pressure and is unaffected by the component in phase with the velocity. Therefore it is affected by β but not by α.

(d) p is zero at both ends of the tube, which means that there are no pressure fluctuations there and therefore no heat release fluctuations caused by pressure fluctuations. Therefore b is zero for $x_f = 0$ or 1.

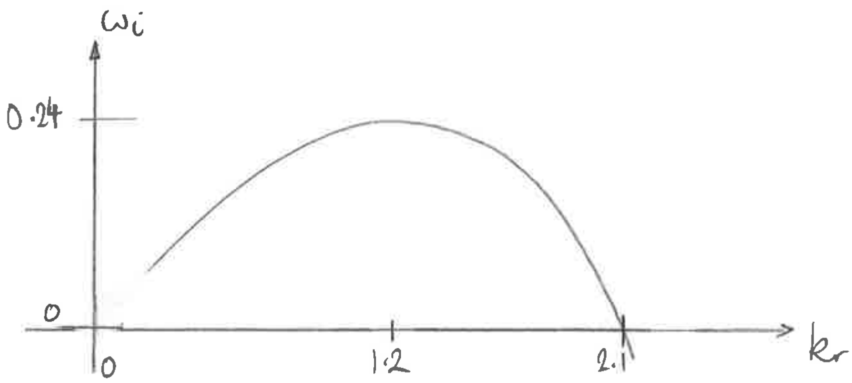
u is zero at the centre of the tube, which means that there are no velocity fluctuations there and therefore no heat release fluctuations caused by velocity fluctuations. Therefore a is zero for $x_f = 0.5$. a is also zero at $x=0$ and $x=1$, even though the velocity fluctuations are maximal there, because p is zero there, and the pressure needs to vary at the flame position in order for work to be done on the gas around the wire.

4.

(a) Velocity fluctuations are of the form:

$u(z) \exp(i(kx - \omega t))$

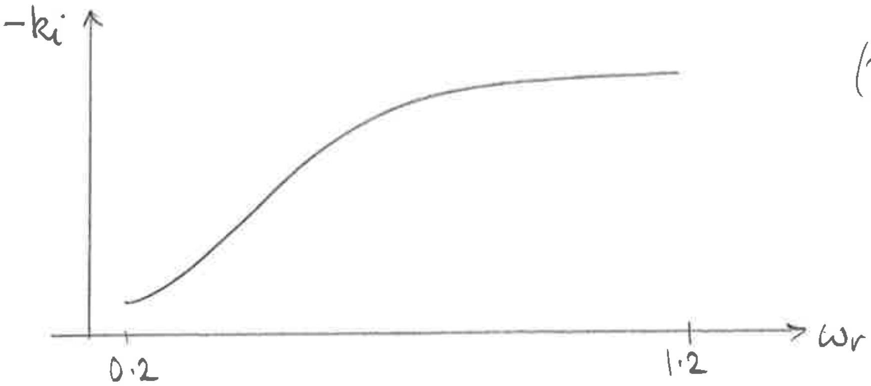
i)



ii) The group velocity is zero at the saddle point: $k_0 = 1.25 - 0.7i$

$\omega_{0i} \simeq 0.14i$ at the saddle point. This is greater than zero so the flow is absolutely unstable

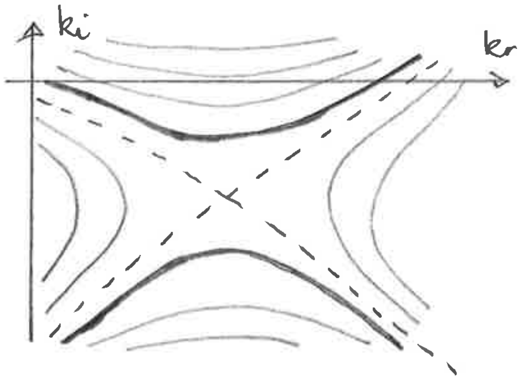
iii) Follow the contour $\omega_i = 0$ and plot $-k_i$ as a function of ω_r:



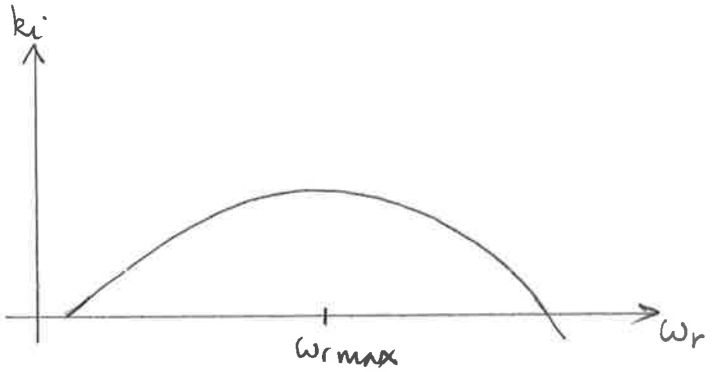
(There is no peak)

A spatial stability analysis has no physical meaning here because the flow is absolutely unstable. In a convectively unstable flow, the spatial analysis measures the spatial growth rate of a time-harmonic signal applied at a point in space. In an absolutely unstable flow, the intrinsic oscillation of the flow grows without limit and drowns out the response to the extrinsic signal.

(b) i)

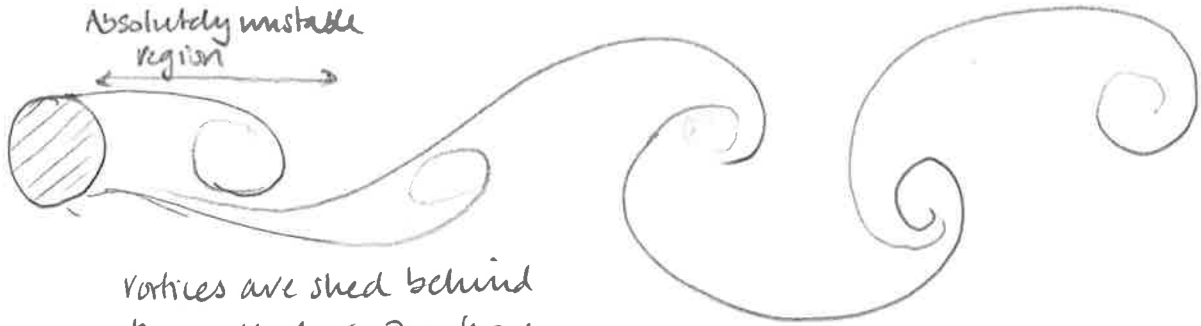


ii)



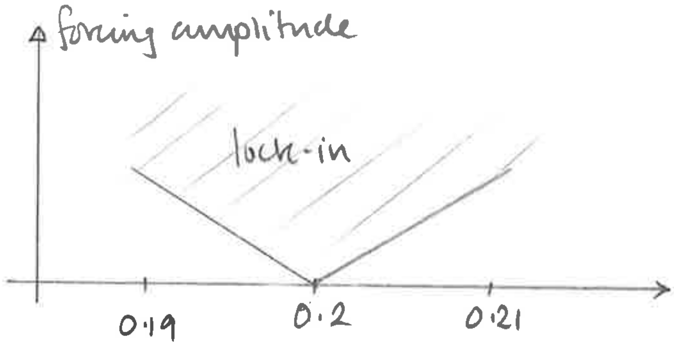
The flow is convectively unstable at $x = 5$. This means that, when considering only this slice of the flow, the intrinsic oscillation decays, but the flow amplifies extrinsic disturbances. Therefore the spatial growth rate does have physical meaning.

(c) Re = 100



Vortices are shed behind the cylinder. The flow oscillates up and down just behind the cylinder.

(d)



When the forcing amplitude exceeds a given amplitude (which depends on the forcing frequency) the frequency of vortex shedding locks into the forcing frequency. The lock-in amplitude increases approximately linearly with $|f_f - f_n|$ where f_f is the forcing frequency and f_n is the natural frequency.

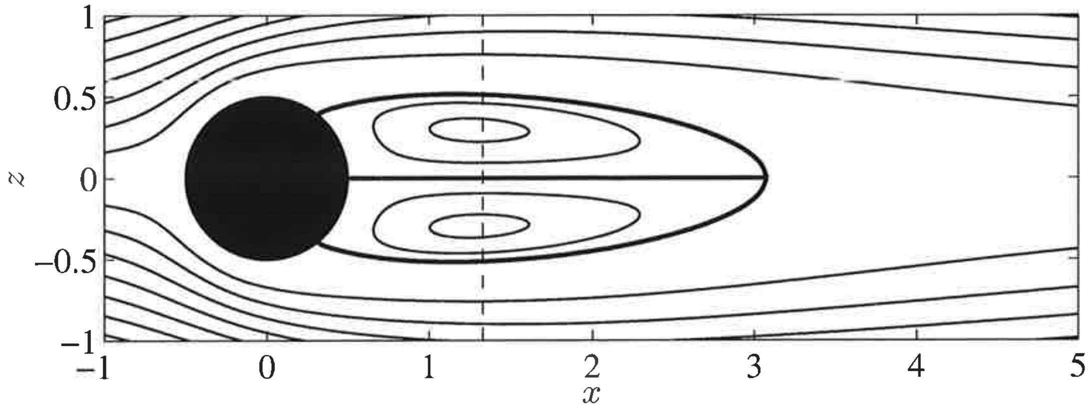

z
x

Fig. 3

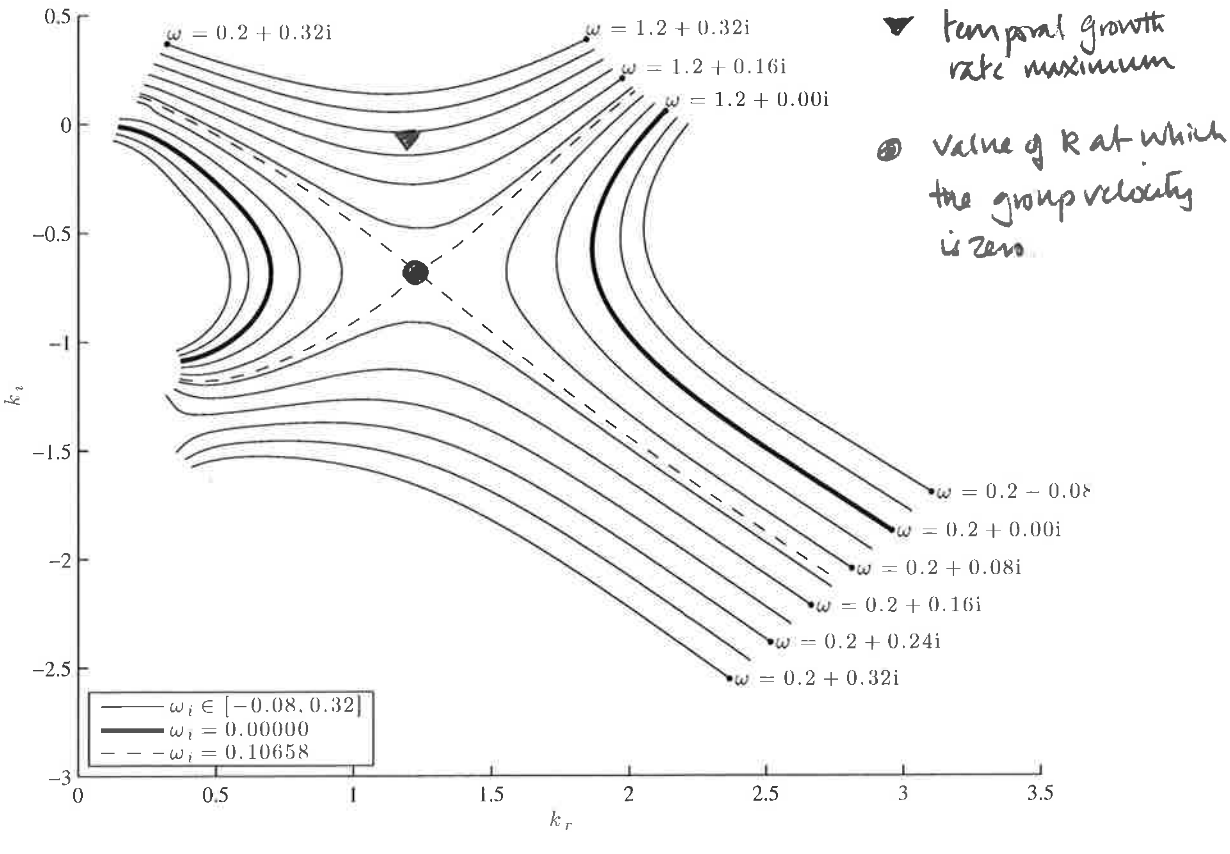

ω = 0.2 + 0.32i
ω = 1.2 + 0.32i
ω = 1.2 + 0.16i
ω = 1.2 + 0.00i
ω = 0.2 − 0.0
ω = 0.2 + 0.00i
ω = 0.2 + 0.08i
ω = 0.2 + 0.16i
ω = 0.2 + 0.24i
ω = 0.2 + 0.32i
ω_i ∈ [−0.08, 0.32]
ω_i = 0.00000
ω_i = 0.10658
k_i
k_r
temporal growth rate maximum
value of k at which the group velocity is zero

Fig. 4