1 (a) Helmholtz's laws:

- The fluid elements that lie on a vortex line at some initial instant continue to lie on that line for all time.

- The flux of vorticity, $\Phi = \int \underset{\sim}{\omega} \cdot d\underset{\sim}{S}$, is the same at all cross-sections of a vortex tube and is independent of time.

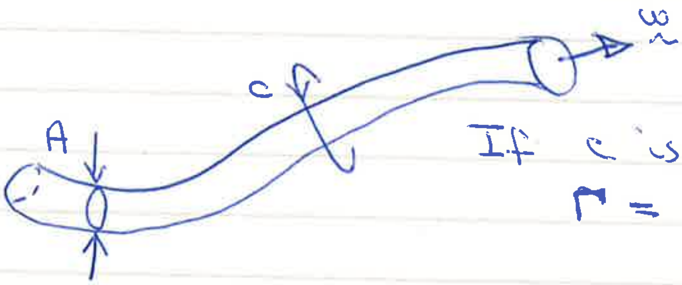
In an inviscid fluid

$$\frac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega} \cdot \nabla)\underset{\sim}{u}$$

Compare $\quad \frac{D}{Dt}(\delta \underset{\sim}{r}) = (d\underset{\sim}{r} \cdot \nabla)\underset{\sim}{u}$

If at $t=0$ $\quad \underset{\sim}{\omega} = d\underset{\sim}{r}$, then they both behave in the same way, so $\underset{\sim}{\omega}$ and $d\underset{\sim}{r}$ remain coincident.

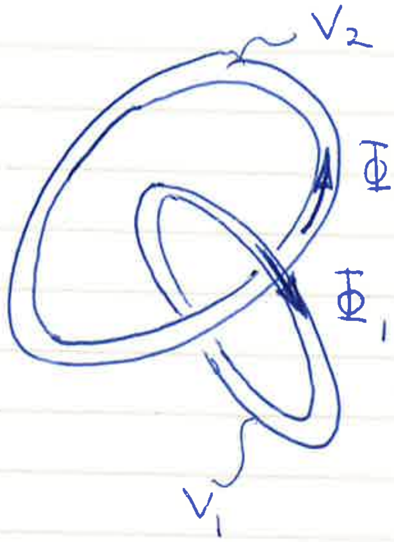
(b) If C is a closed material curve in the fluid, then the circulation $\Gamma = \oint_C \underset{\sim}{u} \cdot d\underset{\sim}{l}$ is independent of time.



If c is a material curve then
$\Gamma = \Phi = \text{const.}$ (Kelvin)
(from Stokes theorem)

So C must always encircle the vortex tube. This is true of <u>all</u> material curves which encircle the tube at $t=0$, which is possible only if the tube itself moves with the fluid. We recover Helmholtz 1 if $A \to 0$.

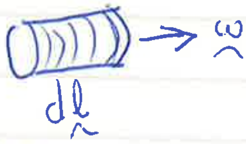
(c)



$$H = \int \underset{\sim}{u} \cdot \underset{\sim}{\omega} \, dV$$

$$= \int_{V_1} \underset{\sim}{u} \cdot \underset{\sim}{\omega} \, dV + \int_{V_2} \underset{\sim}{u} \cdot \underset{\sim}{\omega} \, dV$$

For thin tubes $\underset{\sim}{\omega} \, dV = \Phi \, d\underset{\sim}{l}$



$$\Rightarrow \quad H = \oint_{C_1} \Phi_1 \, \underset{\sim}{u} \cdot d\underset{\sim}{l} + \oint_{C_2} \Phi_2 \, \underset{\sim}{u} \cdot d\underset{\sim}{l} = \Phi_1 \oint_{C_1} \underset{\sim}{u} \cdot d\underset{\sim}{l} + \Phi_2 \oint_{C_2} \underset{\sim}{u} \cdot d\underset{\sim}{l}$$

(Since Φ is constant along vortex tube)

Stokes' theorem has
$$\begin{cases} \oint_{C_1} \underset{\sim}{u} \cdot d\underset{\sim}{l} = \pm \Phi_2, & \text{if tubes linked} \\ \oint_{C_2} \underset{\sim}{u} \cdot d\underset{\sim}{l} = \pm \Phi_1, & \text{if tubes linked} \end{cases}$$

(+ sign for right-handed linkage)

$$\Rightarrow \quad H = \pm \Phi_1 \Phi_2 \pm \Phi_1 \Phi_2 = \pm 2 \Phi_1 \Phi_2 \quad \text{(linked tubes)}$$

If tubes NOT linked, Stokes gives $\oint_{C_1} \underset{\sim}{u} \cdot d\underset{\sim}{l} = \oint_{C_2} \underset{\sim}{u} \cdot d\underset{\sim}{l} = 0$,
and so $H = 0$.

(d) If the fluid is inviscid, then the vortex tubes are locked into the fluid an so linked tubes stay linked and unlinked tubes stay unlinked (Helmholtz # 1)

Also Helmholtz # 2 gives $\Phi_1 = \text{const}$, $\Phi_2 = \text{const}$.

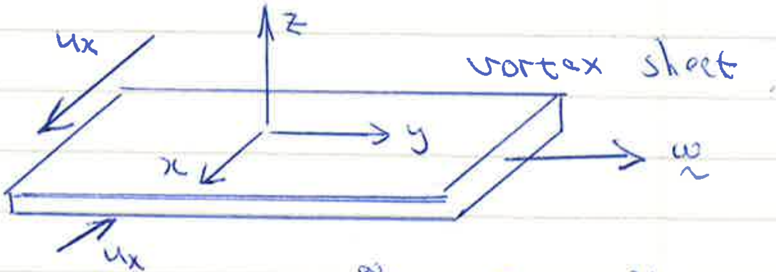
This ensures $H = \text{const}$.

2 (a) $$\frac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u} + \nu\nabla^2\underset{\sim}{\omega}$$

~~advection of vorticity~~ generation of vorticity by vortex line stretching; diffusion of vorticity

Steady soln. requires advection, stretching and diffusion of vorticity to balance.

(b)



(i) Flux (per unit length) $= \int_{-\infty}^{\infty} \omega_y \, dz = \int_{-\infty}^{\infty} \frac{\Phi}{\sqrt{\pi}\, w} e^{-z^2/w^2} dz$

$$= \frac{\Phi}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds, \quad s = \frac{z}{w}$$

$$= \Phi$$

(ii) $\underset{\sim}{u}_{SF} = (0, \alpha y, -\alpha z)$, $\underset{\sim}{u}_{VS} = u_x(z)\,\hat{\underset{\sim}{e}}_x$

$$(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{\omega} = u_z \frac{\partial \underset{\sim}{\omega}}{\partial z} = (-\alpha z)\left(\frac{-2z}{w^2}\right)\underset{\sim}{\omega} = 2\alpha\frac{z^2}{w^2}\underset{\sim}{\omega}$$

$$(\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u} = \omega \frac{\partial u_y}{\partial y}\hat{\underset{\sim}{e}}_y = \alpha\underset{\sim}{\omega}$$

$$\nabla^2\underset{\sim}{\omega} = \frac{\partial^2}{\partial z^2}\underset{\sim}{\omega} = \frac{\partial}{\partial z}\left(-\frac{2z}{w^2}\right)\underset{\sim}{\omega} = -\frac{2}{w^2}\underset{\sim}{\omega} + \left(\frac{2z}{w^2}\right)^2\underset{\sim}{\omega}$$

For steady soln, $(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{\omega} - (\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u} = \nu\nabla^2\underset{\sim}{\omega}$

$$\Rightarrow \quad 2\alpha\frac{z^2}{w^2}\underset{\sim}{\omega} - \alpha\underset{\sim}{\omega} = \nu\left[-\frac{2}{w^2}\underset{\sim}{\omega} + \frac{4z^2}{w^4}\underset{\sim}{\omega}\right]$$

$$\Rightarrow \quad \left(\frac{2z^2}{w^2} - 1\right)\alpha\underset{\sim}{\omega} = \left(\frac{2z^2}{w^2} - 1\right)\frac{2\nu}{w^2}\underset{\sim}{\omega}$$

LHS = RHS provided $\underline{\underline{\alpha = 2\nu / w^2}}$

(iii) For $\alpha = 2\nu/w^2$ there is a balance of advection inward to the $z=0$ plane, outward diffusion and vortex generation by stretching.

If $\alpha > 2\nu/w^2$, advection and stretching will outweigh diffusion, so the vortex sheets thins.

If $\alpha < 2\nu/w^2$ advection and stretching are reduced relative to diffusion, so the vortex sheet thickens.

3

$U = 10$ m/s $\quad u = 1$ m/s $\quad L_t = 0.1$ m

$\nu = 1.5 \times 10^{-5}$ m²/s (air at ambient conditions)

(a) $Re_t = \dfrac{u L_t}{\nu} = \underline{\underline{6667}}$

$Re_\lambda = \dfrac{u\lambda}{\nu} \qquad \varepsilon = 15 \nu\, u^2/\lambda^2$

But also $\qquad \varepsilon = \dfrac{u^3}{L_t} \Rightarrow \dfrac{\lambda^2}{L^2} = 15 \dfrac{\nu}{uL}$

$\Rightarrow \lambda = L \sqrt{15}\, Re_t^{-1/2}$

$\Rightarrow Re_\lambda = \dfrac{u\lambda}{\nu} = \sqrt{15}\, Re_t^{1/2}$

$\Rightarrow \underline{\underline{Re_\lambda = 316}}$

$\eta_k = L_t Re_t^{-3/4} = 0.14$ mm

$\tau_k = \dfrac{L_t}{u} Re_t^{-1/2} = 1.22$ ms

(b) Fastest frequency: $\dfrac{U}{\eta_k} = \dfrac{\text{mean velocity}}{\text{smallest scale}}$

$= 71.4$ kHz

$T_{integral}$ is <u>not</u> turnover time, but

it is $\dfrac{L_t}{U} = 10$ ms

(Turnover time (or eddy lifetime) is $\dfrac{L_t}{u} = 100$ ms)

(c) In the limit of weak turbulence, i.e. if $\frac{u}{\bar{U}} \ll 1$, the turbulence as it passes past a probe can be considered as "frozen", which implies $\frac{\partial}{\partial t} \simeq \bar{U}\frac{\partial}{\partial x}$

Therefore quantities that are based on small-scale gradients (such as $\frac{\partial u_i}{\partial x}$ or $\frac{\partial \phi}{\partial x}$) can be measured by following the temporal gradients. This, for example, allows the measurement of ε based on a measurement of $\frac{\partial u}{\partial t}$.

For the present flow, $\frac{u}{U} = 0.1$, so Taylor hypothesis is valid.

4

(a)

$$P_{11} = -\Big[\overline{u_1u_1}\frac{\partial U_1}{\partial x_1} + \overline{u_1u_2}\frac{\partial U_1}{\partial x_2} + \overline{u_1u_3}\frac{\partial U_1}{\partial x_3} \quad \left(\text{the 3 terms from } 1^{st} \text{ term}\right)$$

$(i=1, j=1)$

$$+ \overline{u_1u_1}\frac{\partial U_1}{\partial x_1} + \overline{u_1u_2}\frac{\partial U_1}{\partial x_2} + \overline{u_1u_3}\frac{\partial U_1}{\partial x_3}\Big] \quad \left(\text{the 3 terms from } 2^{nd} \text{ term}\right)$$

$\frac{\partial U_1}{\partial x_3} = 0$ (2-D flow); $\frac{\partial U_1}{\partial x_2} = 0$ (since $U_1 = Ax_1$)

$$\frac{\partial U_1}{\partial x_1} = A \implies P_{11} = -2\,\overline{u_1^2}\,A$$

Similarly, P_{33} is evaluated by setting $i=3, j=3$ & summing over k $\implies P_{33} = 0$

Similarly, $P_{22} = -\Big[\overline{u_2u_1}\frac{\partial U_2}{\partial x_1} + \overline{u_2u_2}\frac{\partial U_2}{\partial x_2} + \overline{u_2u_3}\frac{\partial U_2}{\partial x_3}$

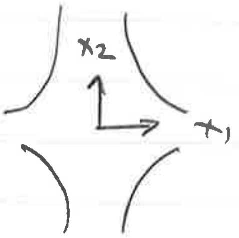
$(i=2, j=2)$

$$+ \overline{u_2u_1}\frac{\partial U_2}{\partial x_1} + \overline{u_2u_2}\frac{\partial U_2}{\partial x_2} + \overline{u_2u_3}\frac{\partial U_2}{\partial x_3}\Big]$$

$$= -2\,\overline{u_2^2}\frac{\partial U_2}{\partial x_2}, \quad \text{but } \frac{\partial U_2}{\partial x_2} = -A$$

$$= 2A\,\overline{u_2^2}$$

Therefore:



the u_2 component is amplified
the u_1 component is reduced
(the u_3 is fed from pressureterms)

(b) For turbulent kinetic energy, we can approach it in two ways:

- $k = \frac{1}{2}\left(\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}\right)$

$$\Rightarrow P = \tfrac{1}{2}(P_{11} + P_{22} + P_{33})$$

$$= A\left(\overline{u_2^2} - \overline{u_1^2}\right)$$

using results from part (a)

- Using $P = -\overline{u_i u_j}\dfrac{\partial U_i}{\partial x_j}$

expanding & summing, and considering that the only terms remaining are those with $\dfrac{\partial U_1}{\partial x_1}$ and $\dfrac{\partial U_2}{\partial x_2}$, we get

$$P = -\left(\overline{u_1^2}\frac{\partial U_1}{\partial x_1} + \overline{u_2^2}\frac{\partial U_2}{\partial x_2}\right)$$

Since $\dfrac{\partial U_1}{\partial x_1} = A$ and $\dfrac{\partial U_2}{\partial x_2} = -A$,

$$P = A\left(\overline{u_2^2} - \overline{u_1^2}\right)$$

(c) From Data Card, $P = \frac{1}{2}\nu_t\left(\dfrac{\partial U_i}{\partial x_j} + \dfrac{\partial U_j}{\partial x_i}\right)^2$

ν_t = turbulent viscosity

Since $U_3 = 0$, $\dfrac{\partial U_1}{\partial x_2} = \dfrac{\partial U_1}{\partial x_3} = 0$ & $\dfrac{\partial U_2}{\partial x_1} = \dfrac{\partial U_2}{\partial x_3} = 0$,

$$P = \frac{1}{2}\nu_t\left[\underbrace{\left(\frac{\partial U_1}{\partial x_1} + \frac{\partial U_1}{\partial x_1}\right)}_{2A}^2 + \underbrace{\left(\frac{\partial U_2}{\partial x_2} + \frac{\partial U_2}{\partial x_2}\right)}_{-2A}^2\right] = 4\nu_t A^2$$

Compare with Part (b): P is $A(\overline{u_2^2}-\overline{u_1^2})$

The k-ε model gives $P = 4\nu_t A^2$

Obviously, the real production term could be -ve, +ve or zero, depending on anisotropy. Instead, k-ε gives P +ve always. The k-ε is mostly meant for shear-type flows, where the turbulence generation is dominated by shear. In this stagnation flow, turbulence generation is dominated by normal stresses and mean velocity strain. In practice, there are corrections to the k-ε model to make it approximately OK for more complex flows, although in principle, for such flows the full Re-stress solution is necessary.