1.

(a) $$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \underline{v}' = 0 \qquad -(1)$$

$$\rho_0 \frac{\partial \underline{v}'}{\partial t} + \rho_0 \alpha \underline{v}' = -\nabla p' \qquad -(2)$$

Taking $\frac{\partial}{\partial t}$ of (1) & $\nabla\cdot$ (2) & subtracting, we get,

$$\frac{\partial^2 \rho'}{\partial t^2} - \rho_0 \alpha \nabla \cdot \underline{v}' = \nabla^2 p'$$

From eq^n (1), $\rho_0 \nabla \cdot \underline{v}' = -\frac{\partial \rho'}{\partial t}$

$$\therefore \quad \frac{\partial^2 \rho'}{\partial t^2} + \alpha \frac{\partial \rho'}{\partial t} - \nabla^2 p' = 0.$$

$$p' = c_0^2 \rho'$$

$$\Rightarrow \quad \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} + \frac{\alpha}{c_0^2} \frac{\partial p'}{\partial t} - \nabla^2 p' = 0$$

or $$\boxed{\frac{\partial^2 p'}{\partial t^2} + \alpha \frac{\partial p'}{\partial t} - c_0^2 \nabla^2 p' = 0} \qquad -(3)$$

(b) $p' = A e^{i\omega t - i \underline{k} \cdot \underline{x}}$

↳ in Eq (3)

$$-\omega^2 + i\omega\alpha + c_0^2 k^2 = 0 .$$

$$\Rightarrow k = \sqrt{\frac{\omega^2}{c_0^2} - \frac{i\omega\alpha}{c_0^2}}$$

$$k = \frac{\omega}{c_0}\sqrt{1 - \frac{i\alpha}{\omega}}$$

if $\alpha \ll 1$

$$k \sim \frac{\omega}{c_0}\left(1 - \frac{i}{2}\frac{\alpha}{\omega}\right)$$

$$\boxed{k \sim \frac{\omega}{c_0} - \frac{i\alpha}{2c_0}}$$

(c) $\underline{v}' \cdot$ (Eq (2)) gives

$$\frac{1}{2}\rho_0 \frac{\partial v'^2}{\partial t^2} + \rho_0 \alpha v'^2 = -\underline{v}' \cdot \nabla p' \qquad -(4)$$

$\frac{1}{\rho_0} p'$ (Eq (1)) gives:

$$\frac{1}{\rho_0} p' \frac{\partial p'}{\partial t} + p' \nabla \cdot \underline{v}' = 0 \qquad -(5)$$

Eq (4) + (5) gives

$$\frac{1}{2}\rho_0 \frac{\partial v'^2}{\partial t^2} + \rho_0 \alpha v'^2 + \frac{1}{\rho_0 c_0^2} p' \frac{\partial p'}{\partial t} + \nabla \cdot (p' \underline{v}') = 0$$

$$\Rightarrow \frac{1}{2}\rho_0 \frac{\partial v'^2}{\partial t} + \rho_0 \alpha v'^2 + \frac{1}{2\rho_0 c_0^2} \frac{\partial p'^2}{\partial t} + \nabla \cdot (p' \underline{v}') = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left\{ \frac{1}{2}\rho_0 v'^2 + \frac{1}{2}\frac{p'^2}{\rho_0 c_0^2} \right\} + \nabla \cdot (p' \underline{v}') = -\rho_0 \alpha v'^2$$

$$\Rightarrow w = \frac{1}{2}\rho_0 v'^2 + \frac{1}{2}\frac{p'^2}{\rho_0 c_0^2}$$

$$\underline{I} = p' \underline{v}'$$

$$\mathcal{D} = \rho_0 \alpha v'^2$$

2(a) Using the Green's function from the data sheet,

$$\rho'(\underline{x},t) = -\iint \frac{\partial F_i}{\partial y_i} \frac{\delta\{|\underline{x}-\underline{y}| - c_0(t-\tau)\}}{4\pi c_0 |\underline{x}-\underline{y}|} \, d\tau \, d\underline{y} \quad -(1)$$

Integrating with respect to τ (by using the integration property of a Delta function), we get

$$\rho'(\underline{x},t) = -\frac{1}{4\pi c_0^2} \int \frac{\partial F_i}{\partial y_i}\left(\underline{y}, t - \frac{|\underline{x}-\underline{y}|}{c_0}\right) \frac{d\underline{y}}{|\underline{x}-\underline{y}|} \quad -(2)$$

- Far field assumption $\frac{1}{|\underline{x}-\underline{y}|} \sim \frac{1}{|\underline{x}|} = \frac{1}{x}$
- For a compact source, $t - \frac{|\underline{x}-\underline{y}|}{c_0} \sim t - \frac{x}{c_0}$

$\therefore$ Eq (2) becomes.

$$\rho'(x,t) \sim -\frac{1}{4\pi c_0^2 x} \int \frac{\partial F_i}{\partial y_i}\left(\underline{y}, t - \frac{x}{c_0}\right) d\underline{y}$$

Using the hint, we get.

$$\rho'(x,t) = \frac{1}{4\pi c_0^3 x} \underbrace{\left(\frac{x_i}{x}\right)}_{\beta_i} \int \frac{\partial F_i}{\partial t}\left(\underline{y}, t - \frac{x}{c_0}\right) d\underline{y}$$

Now making the estimates,

$$F \sim \frac{\rho_0 u'^2}{l}, \quad \frac{\partial}{\partial t} \sim \frac{u'}{l}, \quad \int d\underline{y} \sim l^3$$

We get,

$$\boxed{p' \sim \rho_0 \beta_i \left(\frac{l}{x}\right) m^3}$$

(b) The acoustic intensity in the far field is given by

$$I = p'u' = \frac{p'^2}{\rho_0 c_0} = \frac{c_0^3 \rho'^2}{\rho_0} \quad (\text{using } p' = c_0^2 \rho')$$

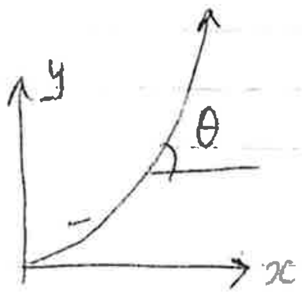
$$\therefore \; I \sim \rho_0 c_0^3 (\beta_i)^2 \left(\frac{l}{x}\right)^2 m^6$$

Power scales as $I * x^2$

$$\therefore \; \boxed{P \sim \rho_0 c_0^3 \beta_i^2 l^2 m^6}$$

3.

(a)



Snell's Law: $\dfrac{\sin\theta}{c_0} = $ constant on ray.

$\tan\theta = y' \Rightarrow \sin\theta = \dfrac{y'}{(1+y'^2)^{1/2}}$

$$\therefore \quad \frac{y'}{(\alpha x+\beta)(1+y'^2)^{1/2}} = \frac{\sin\theta_0}{\beta} \leftarrow \text{evaluated at origin.}$$

$$y'^2 = \frac{(\alpha x+\beta)^2 \sin^2\theta_0}{\beta^2}\left[1+y'^2\right]$$

$$\longrightarrow \quad y'^2 = \frac{\dfrac{(\alpha x+\beta)^2\sin^2\theta_0}{\beta^2}}{1 - \dfrac{(\alpha x+\beta)^2\sin^2\theta_0}{\beta^2}}$$

$$y' = \frac{\dfrac{(\alpha x+\beta)\sin\theta_0}{\beta}}{\left[1-\dfrac{(\alpha x+\beta)^2\sin^2\theta_0}{\beta^2}\right]^{1/2}}$$

(choose + root to match diagram)

$$\int dy = \int dx \, \frac{\dfrac{(\alpha x+\beta)\sin\theta_0}{\beta}}{\left[1-\dfrac{(\alpha x+\beta)^2\sin^2\theta_0}{\beta^2}\right]^{1/2}}$$

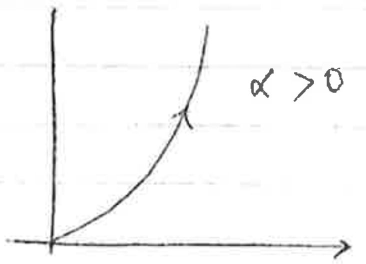
$$y = -\frac{\beta}{\alpha\sin\theta_0}\left[1-\frac{(\alpha x+\beta)^2\sin^2\theta_0}{\beta^2}\right]^{1/2} + k$$

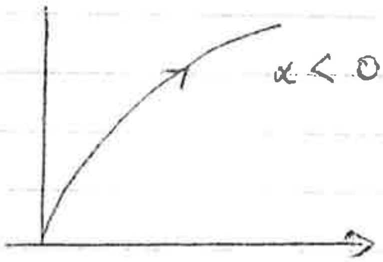
but, ray goes through $x=0, y=0$

$$\Rightarrow \quad k = +\frac{\beta}{\alpha\sin\theta_0}(1-\sin^2\theta_0)^{1/2} = \frac{\beta\cot\theta_0}{\alpha}$$

$$\left(y - \frac{\beta\cot\theta_0}{\alpha}\right)^2 = \frac{\beta^2}{\alpha^2\sin^2\theta_0} - \frac{(\alpha x+\beta)^2}{\alpha^2}$$

$$\underline{\left(x + \beta/\alpha\right)^2 + \left(y - \frac{\beta\cot\theta_0}{\alpha}\right)^2 = \frac{\beta^2}{\alpha^2\sin^2\theta_0}}$$



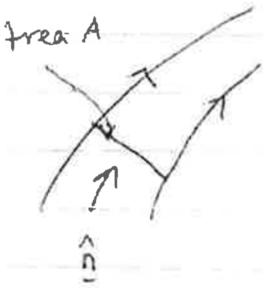


Circle centre $\left(\frac{-\beta}{\alpha}, \frac{\beta\cot\theta_0}{\alpha}\right)$, radius $\frac{\beta}{|\alpha|\sin\theta_0}$

If $\alpha = 0$ then straight line at angle θ_0

$$\underline{y = \tan\theta_0\, x.}$$

Consider adjacent rays, forming ray tube [40]



Energy flux along tube $= \underset{\sim}{I} \cdot \hat{\underline{n}}\, A = p' \underset{\sim}{u}' \cdot \hat{\underline{n}}\, A$

Since rays parallel to tube, energy flux conserved $A\, p' \underset{\sim}{u}' \cdot \hat{\underline{n}} = \text{constant}$

Assuming wavelength << variation of medium, have locally plane waves $\Rightarrow$ $\underset{\sim}{u}' \cdot \hat{\underline{n}} = \frac{p'}{\rho_0 c_0}$ [10]

$$\underline{A \frac{p'^2}{\rho_0 c_0} = \text{constant}}$$

Qu 4.

a) An acoustic mode in a duct is said to be 'cut-off' if it is evanescent. Then the axial wavenumber is purely imaginary so that the amplitude of the disturbance decays exponentially along the duct.

b) The formula on the data card gives the pressure field as

$$p'(\underset{\sim}{x},t) = e^{i(\omega t + n\theta)} J_n\left(\frac{z_{mn} r}{a}\right)\left(Ae^{-ikx_3} + Be^{ikx_3}\right)$$

where z_{mn} is the mth zero of $\dot{J}_n(z)$ and $k = (k_0^2 - z_{mn}^2/a^2)^{1/2}$, $k_0 = \omega/c_0$.

We denote the rotation rate by Ω, blade passing frequency is 3Ω.

The flow field is steady in the rotating frame, i.e. $p'(\underset{\sim}{x},t)$ is a function of $(\omega t - \theta)$

At harmonics of the blade passing frequency $\omega = 3M\Omega$, M integer
$n = -3M$

For cut-off modes we need k imaginary i.e. $k_0 < \frac{z_{mn}}{a}$

$$\frac{3M\Omega}{c_0} < \frac{z_{m\,3M}}{a}$$

z_{m3M} increases with m so need $\frac{3M\Omega}{c_0} < \frac{z_{1\,3M}}{a}$ for all M.

For M = 1, we require $\Omega < \frac{c_0}{3a} z_{13} = \frac{c_0}{a}\frac{4.20119}{3}$

$= 2$ $\quad \Omega < \frac{c_0}{3a}\frac{z_{16}}{2} = \frac{c_0}{a}\frac{7.50127}{6}$

$\geqslant 3$ $\quad \Omega < \frac{c_0}{3a}\frac{z_{1\,3M}}{M} = \frac{c_0}{a}\frac{1}{3M}\left(3M + 0.8061\,(3M)^{1/3}\right)$

$$= \frac{c_0}{a}\left(1 + 0.8061\,(3M)^{-2/3}\right)$$

It is clear that the most stringent constraint is for large M and that we need $\Omega < c_0/a = \frac{343}{0.15} = 2.2867 \times 10^3$ rads^{-1}

$= \underline{\underline{21,836 \text{ rpm}}}$

c) The rotor pattern has the form $e^{iM3(\Omega t - \theta)}$

and the stator pattern is $e^{\mp iNS\theta}$ where S = number of stator blades.

So rotor-stator interaction modes are: $e^{iM3(\Omega t-\theta) \mp iNS\theta}$

$$= e^{i(3M\Omega t - (M3 \pm NS)\theta)}$$

Modes propagate if $\dfrac{3M\Omega a}{c_0} > z_{1\ 3M \pm NS}$

Usually $S \simeq 2.4 \times$ number of rotor blades and S prime is a good choice. Here $2.4 \times 3 = 7.2$ so try $S = 7$.

For 10,000 rpm $\dfrac{\Omega a}{c_0} = 0.458 < 1$ so all rotor-alone modes are cut-off.

Modes of blade passing frequency propagate if

$$0.458 \times 3 > z_{1|3 \mp N7|}$$

i.e if $0.458 > \dfrac{z_{1|3 \mp N7|}}{3}$

$\dfrac{z_{1|3 \mp N7|}}{3} > 1$ for all N and so certainly > 0.458.

<u>All modes at bpf are cut-off.</u>