4C7 Random and Non-linear Vibrations 2013 Crib

1\. a) $\ddot{\beta} + \alpha\Omega\dot{\beta} + \Omega^2\beta = \gamma\Omega V(t)$

$\downarrow$

approximate to white noise with
double-sided spectrum $S_{vv}(\Omega) \Rightarrow$ spectrum of right hand side $= \gamma^2\Omega^2 S_{vv}(\Omega)$

White noise results: $\sigma_\beta^2 = \dfrac{\pi\gamma^2\Omega^2 S_{vv}(\Omega)}{\alpha\Omega \times \Omega^2} = \underline{\dfrac{\pi\gamma^2 S_{vv}(\Omega)}{\alpha\Omega}}$

$\sigma_{\dot{\beta}}^2 = \Omega^2\sigma_\beta^2 = \underline{1/\alpha\left[\pi\gamma^2 S_{vv}(\Omega)\Omega\right]}$

Rate of crossing b with +ve slope $\nu_b^+ = \dfrac{1}{2\pi}\left(\dfrac{\sigma_{\dot{\beta}}}{\sigma_\beta}\right) e^{-\frac{1}{2}(b/\sigma_\beta)^2} = \underline{\dfrac{\Omega}{2\pi} e^{-b^2\alpha\Omega/(2\pi\gamma^2 S_{vv})}}$

Probability of Failure $P = \underbrace{1 - e^{-\nu_b^+ T}}_{\text{maximum when } \nu_b^+ \text{ is maximum}}$

$\nu_b^+ = \dfrac{\Omega}{2\pi} e^{-z\Omega}$ where $z = b^2\alpha/(2\pi\gamma^2 S_{vv}) = 0.8^2 \times 0.1/(2\pi \times 0.1^2 \times 5) = 0.204$

For Maximum $\dfrac{\partial}{\partial\Omega}\nu_b^+(\Omega) = 0 \Rightarrow (1 - z\Omega)e^{-z\Omega} = 0 \Rightarrow \Omega = 1/z = 4.9$ rad/s

$\Rightarrow \nu_b^+ = \dfrac{4.9}{2\pi} e^{-1} = 0.287$

$P = 1 - e^{-0.287T}$

$\downarrow$

time for 50 revolutions $= \left(\dfrac{2\pi}{\Omega}\right) \times 50 = 64$ secs

$P = 1 - e^{-0.287 \times 64} \approx \underline{1}$ [70%]

b) $E[D(T)] = \nu_0^+ T\, E\left[\dfrac{1}{N(s)}\right] = \nu_0^+ T \int_0^\infty \left(\dfrac{1}{c}\right) s^r p(s)\, ds$

$\downarrow$

Rayleigh distribution $p(s) = \dfrac{s}{\sigma_\beta^2} e^{-\frac{1}{2}(s/\sigma_\beta)^2}$

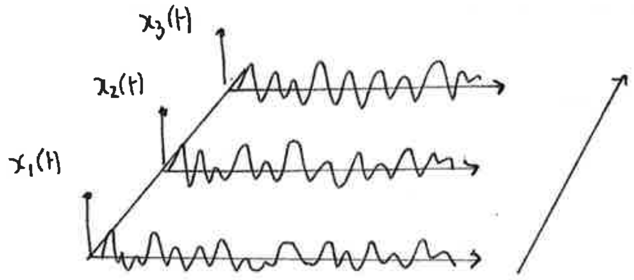
$\Rightarrow \underline{E[D(T)] = \nu_0^+ T \int_0^\infty \left(\dfrac{1}{c}\right) \dfrac{s^{r+1}}{\sigma_\beta^2} e^{-\frac{1}{2}(s/\sigma_\beta)^2} ds}$

Now it can be noted that $E\left[\dfrac{1}{N(s)}\right] \propto E[s^r] \propto \sigma_s^{\,r} \propto \sigma_\beta^{\,r}$; $\sigma_\beta \propto \Omega^{-1/2}$ from part (a)

$\Rightarrow E[D(T)] \propto \Omega \times E\left[\dfrac{1}{N(s)}\right] \propto \Omega \times \Omega^{-r/2} = \underline{\Omega^{1-r/2}} \Rightarrow \underline{n = 1 - r/2}$ [30%]

$\uparrow$ from ν_0^+

2 a).



Ensemble

Each sample (or realisation) has a different set of phase angles ε_n

[15%]

b) $E[x(t)] = \sum_{n=1}^{N} E[\cos(\omega_n t + \varepsilon_n)] = \sum_{n=1}^{N} \int_0^{2\pi} \cos(\omega_n t + \varepsilon_n)\, p(\varepsilon_n)\, d\varepsilon_n = 0$

$\uparrow$ constant $\left(\frac{1}{2\pi}\right)$

For large N, consider temporal average:

$$\langle x(t) \rangle = \sum_{n=1}^{N} \int_{t_0}^{t_0+T} \cos(\omega_n t + \varepsilon_n)\, dt \times \left(\frac{1}{T}\right)$$

$$= -\sum_{n=1}^{N} \left(\frac{1}{\omega_n T}\right) \left[\sin(\omega_n t_0 + \omega_n T + \varepsilon_n) - \sin(\omega_n t_0 + \varepsilon_n)\right] \quad ; \; \omega_n = \frac{n\pi}{T}$$

$\downarrow$ expect summation over n to be zero, if there are enough terms in the series $\sum_{n=1}^{N} \equiv \int dn$

$\Rightarrow$ Sum has the same effect as ensemble average

$\Rightarrow$ For large N, expect process to be ergodic - temporal averages involve summations, summations are like integrals over n, integral over n is same as integral over ε, same as ensemble average.

For $N = 1$ $\quad x(t) = \cos(\omega_1 t + \varepsilon_1) \Rightarrow$ non-ergodic, temporal average on a sample will depend on t.

[30%]

c) $R_{xx}(\tau) = E[x(t)\,x(t+\tau)] = E\left[\sum_n \sum_m \cos(\omega_n t + \varepsilon_n)\cos(\omega_m t + \omega_m \tau + \varepsilon_m)\right]$

$$= \tfrac{1}{2} E\Big[\sum_n \sum_m \cos(\omega_n t + \omega_m t + \omega_m \tau + \varepsilon_n + \varepsilon_m)$$

average always zero

$$+ \cos(\omega_n t - \omega_m t - \omega_m \tau + \varepsilon_n - \varepsilon_m)\Big]$$

average zero unless $n = m$, when average is $\cos(\omega_m \tau)$

$\Rightarrow \underline{R_{xx}(\tau) = \tfrac{1}{2} \sum_n \cos(\omega_n \tau)}$

[30%]

d) $x(t) = \sum_{n=1}^{N} \cos(\omega_n t + \varepsilon_n)$ $\longrightarrow$ <u>central limit theorem</u> implies that $x(t)$ will be <u>Gaussian</u> For lage N

$$\Rightarrow \quad p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

Now $E[x] = 0 \Rightarrow \sigma^2 = E[x^2] = R_{xx}(0) = \frac{1}{2}\sum_{n=1}^{N} 1 = \frac{1}{2}N \Rightarrow \sigma = \sqrt{N/2}$

$$\Rightarrow \quad \underline{p(x) = \frac{1}{\sqrt{\pi N}} e^{-x^2/N}}$$

If N is not large then the central limit theorem does not apply. [25%]

Q3
$$\ddot{x} + p^2 x + \epsilon\alpha_1 x^2 + \epsilon^2(\mu\dot{x} + \beta_1 x^3 + \beta_2 \dot{x}^3) = 0.$$

(a)
$$\omega^2 = p^2 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots\ldots$$
$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots\ldots$$

substituting in the above equation we get (correct to 2nd order in ϵ)

$$(\ddot{x}_0 + \epsilon\ddot{x}_1 + \epsilon^2\ddot{x}_2) + (\omega^2 - \epsilon\omega_1 - \epsilon^2\omega_2)(x_0 + \epsilon x_1 + \epsilon^2 x_2)$$
$$+ \epsilon\alpha_1(x_0 + \epsilon x_1)^2 + \epsilon^2(\mu\dot{x}_0 + \beta_1 x_0^3 + \beta_2\dot{x}_0^3) = 0$$

separating into equations for each order of ϵ:

$$\ddot{x}_0 + \omega^2 x_0 = 0$$
$$\ddot{x}_1 + \omega^2 x_1 = \omega_1 x_0 + \alpha_1 x_0^2$$
$$\ddot{x}_2 + \omega^2 x_2 = \omega_2 x_0 + \omega_1 x_1 + 2\alpha_1 x_0 x_1 + \mu\dot{x}_0 + \beta_1 x_0^3 + \beta_2\dot{x}_0^3$$

(b) 0th order: $x_0 = A\cos\omega t$

1st order: $\ddot{x}_1 + \omega^2 x_1 = \omega_1 A\cos\omega t + \alpha_1 A^2\cos^2\omega t$

$\therefore \omega_1 = 0$ for bounded amplitude solution

and $\ddot{x}_1 + \omega^2 x_1 = \dfrac{\alpha_1 A^2}{2}(1 + \cos 2\omega t)$

$$\therefore x_1 = \frac{A^2}{6\omega^2}(3\alpha_1 - \alpha_1\cos 2\omega t)$$

or solution correct to 1st order:

$$x = A\cos\omega t + \frac{A^2\epsilon\alpha_1}{6\omega^2}(3 - \cos 2\omega t).$$

(c)

$$\ddot{x}_2 + \omega^2 x_2 = \omega_2 x_0 + 2\alpha_1 x_0 x_1 + \mu\dot{x}_0 + \beta_1 x_0^3 + \beta_2 \dot{x}_0^3$$

$$\text{RHS} = \omega_2 A\cos\omega t + 2\alpha_1 A\cos\omega t\left(\frac{A^2\alpha_1}{6\omega^2}\right)(3-\cos 2\omega t)$$

$$+\mu(-A\omega\sin\omega t) + \beta_1 A^3\cos^3\omega t + \beta_2(-A^3\omega^3)\sin^3\omega t$$

$$\tfrac{1}{4}(\cos 3\omega t + 3\cos\omega t) \qquad \tfrac{1}{4}(3\sin\omega t - \sin 3\omega t)$$

For bounded amplitude solution, terms on the RHS in $\cos\omega t$ and $\sin\omega t$ must go to zero.

$$\sin\omega t: \quad -\mu A\omega - \frac{3}{4}\beta_2 A^3\omega^3 = 0$$

$$\therefore \quad A^2 = \frac{-4}{3}\frac{\mu}{\omega^2\beta_2}$$

A limit cycle exists if μ and β_2 are of opposite sign. Limit cycle is either stable or unstable depending on μ.

$$\cos\omega t: \quad \omega_2 A + \cancel{4}\frac{\alpha_1^2 A^3}{\omega^2} - \frac{1}{6}\frac{\alpha_1^2 A^3}{\omega^2} + \frac{3}{4}\beta_1 A^3 = 0$$

$$\therefore \quad \omega_2 = -\frac{5}{6}\frac{\alpha_1^2 A^2}{\omega^2} - \frac{3}{4}\beta_1 A^2$$

$$= -\frac{5}{6}\frac{\alpha_1^2}{\omega^2}\cancel{A}\left(\frac{-4}{3}\frac{\mu}{\omega^2\beta_2}\right) - \frac{3}{4}\beta_1\left(\frac{-4}{3}\frac{\mu}{\omega^2\beta_2}\right)$$

$$= \frac{\mu}{\omega^2\beta_2}\left(\beta_1 + \frac{10}{9}\frac{\alpha_1^2}{\omega^2}\right)$$

Approximating $\omega^2 \sim p^2$ for initial estimate of ω

$$\omega_2 \simeq \frac{\mu}{p^2\beta_2}\left(\beta_1 + \frac{10}{9}\frac{\alpha_1^2}{p^2}\right).$$

Q4 $\ddot{x} + \alpha \dot{x} + \dot{x}^3 + x - x^3 = 0$

(a)
$$\dot{x} = y$$
$$\dot{y} = -x + x^3 - \alpha y - y^3$$

equilibrium points $\dot{x} = \dot{y} = 0$

or $y = 0$ and $x^3 = x$ or $x = 0, \pm 1$.

(b) Consider $x = 0$, $y = 0$. Linearising we ge

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha \end{bmatrix}$$

eigenvalues:
$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\alpha - \lambda \end{vmatrix} = 0$$

$$\lambda^2 + \alpha \lambda + 1 = 0$$

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

For $\alpha \geq 2 \Rightarrow$ both roots real -ve stable node.

$0 < \alpha < 2 \Rightarrow$ complex -ve real part stable focus.

$\alpha = 0 \Rightarrow$ centre.

$-2 < \alpha < 0 \Rightarrow$ complex +ve real part unstable focus.

$\alpha \leq -2 \Rightarrow$ both roots real +ve unstable node.

consider: $x = +1,\ y = 0$.

$\Rightarrow$ $z = x - 1$ & linearise about $y=0$, $z=0$.

$\dot{z} = \dot{x}$

$$\dot{y} = x(x-1)(x+1) - \alpha y - y^3$$
$$\dot{y} = (z+1)z(z+2) - \alpha y - y^3$$
$$\dot{y} = 2z + H.O(z) - \alpha y - y^3$$

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -\alpha \end{bmatrix}$$

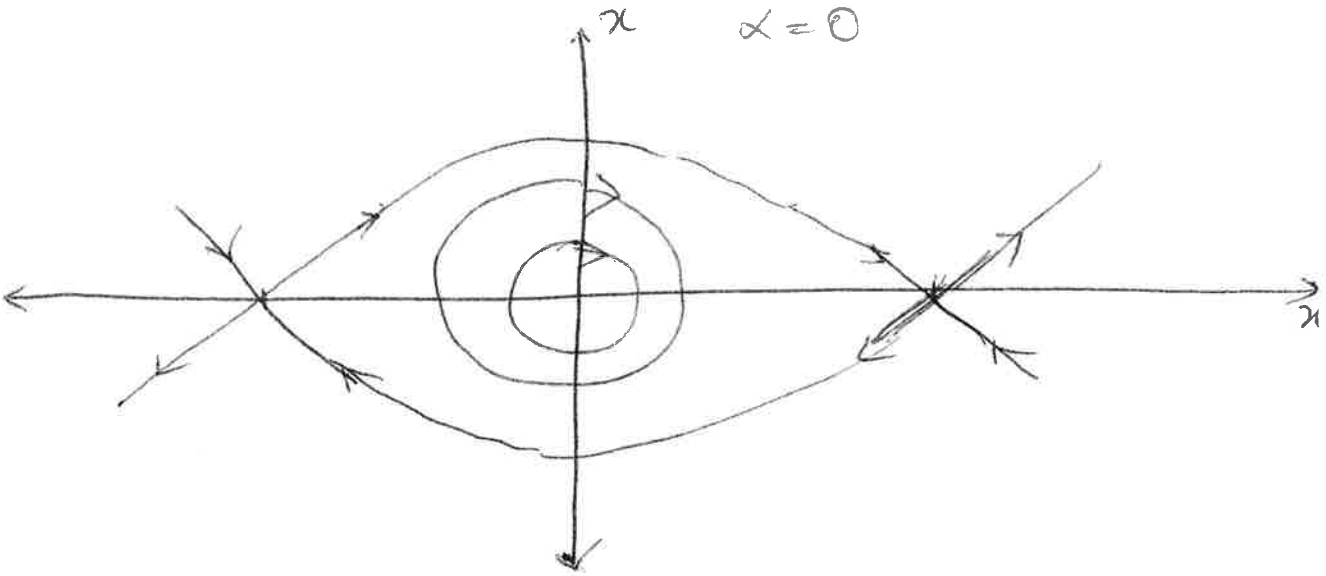
Eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ 2 & -\lambda - \alpha \end{vmatrix} = 0.$$

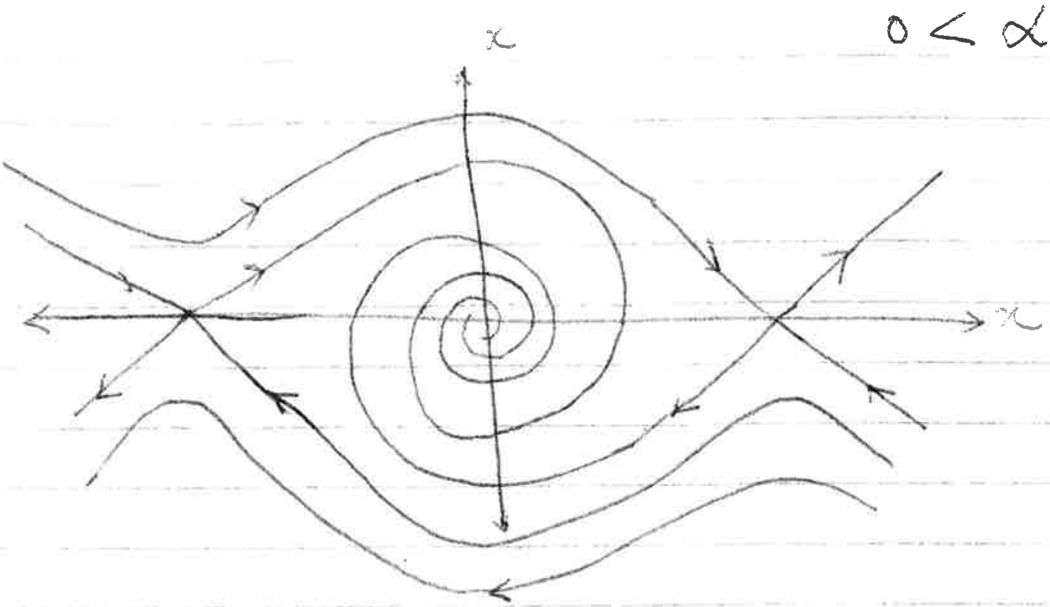
$$\lambda^2 + \alpha\lambda - 2 = 0$$
$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 + 8}}{2}$$

$\alpha \geq 0 \Rightarrow$ saddle point.
$\alpha < 0 \Rightarrow$ saddle point.

(c)

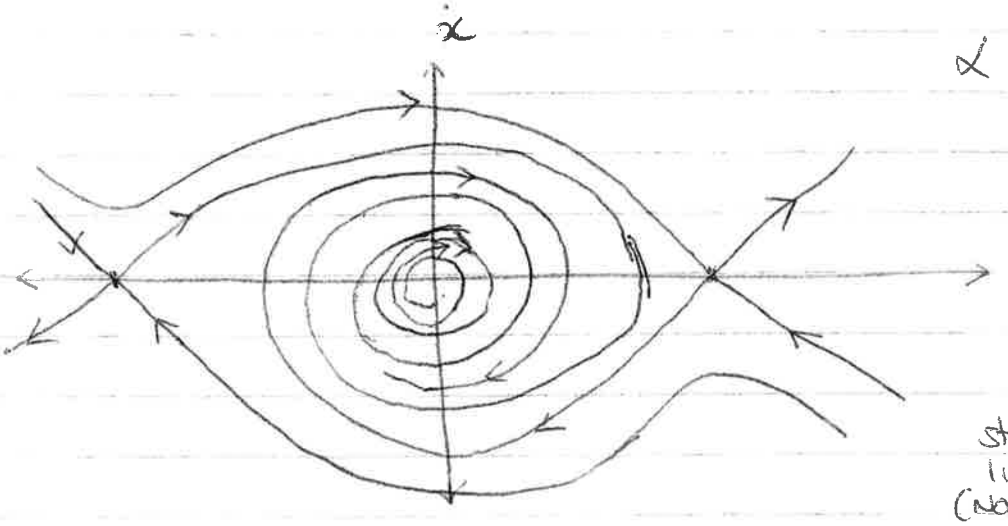


$0 < \alpha < 2$



$x=0$ stable focus

$\alpha < 0$



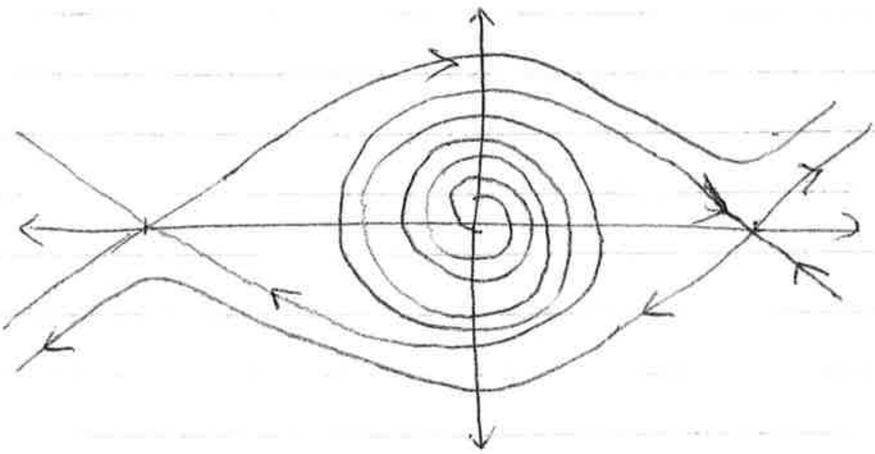
$x=0$ unstable focus

stable limit cycle (NOTE $\dot{x}^3$ term)

$-2 < \alpha < 0$

no limit cycle

$x=0$ unstable focus



(d) Hopf bifurcation @ $x=0$ $\alpha=0$. Transition from stable focus to unstable focus with the system describing transition from decaying amplitude oscillation to unstable growth with an intermediate region showing stable limit cycle oscillation (NOTE $\dot{x}^3$ term).