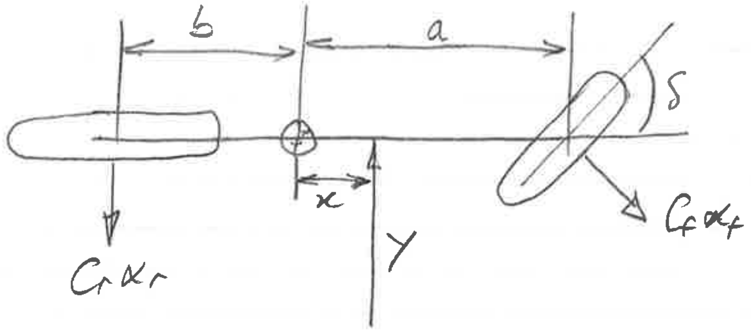
PART IIB ENGINEERING - MODULE 4C8 - 2013

1. (a)



Defns

$C = C_f + C_r$

$s = \dfrac{aC_f - bC_r}{C_f + C_r}$

$q^2 = \dfrac{a^2C_f + b^2C_r}{C_f + C_r}$

$I = mk^2$, $l = a+b$

Equations of motion : Book work - see lecture notes

$$m\begin{bmatrix}1 & 0\\ 0 & k^2\end{bmatrix}\begin{Bmatrix}\dot v\\ \dot\Omega\end{Bmatrix} + \begin{bmatrix}C/u & Cs/u + mu\\ Cs/u & Cq^2/u\end{bmatrix}\begin{Bmatrix}v\\ \Omega\end{Bmatrix} = \begin{Bmatrix}Y + C_f\delta\\ xY + aC_f\delta\end{Bmatrix} \qquad (1)$$

(b) steady state response to side wind

$\dot v = \dot\Omega = 0$ So

$$\begin{bmatrix}C/u & Cs/u + mu\\ Cs/u & Cq^2/u\end{bmatrix}\begin{Bmatrix}v_{ss}\\ \Omega_{ss}\end{Bmatrix} = \begin{Bmatrix}Y + C_f\delta\\ xY + aC_f\delta\end{Bmatrix} \qquad (1a)$$

Solving for unknown steady state velocities :

$$\begin{Bmatrix}v_{ss}\\ \Omega_{ss}\end{Bmatrix} = \frac{\begin{bmatrix}Cq^2/u & -(Cs/u + mu)\\ -Cs/u & C/u\end{bmatrix}\begin{Bmatrix}Y + C_f\delta\\ xY + aC_f\delta\end{Bmatrix}}{\left(\frac{C}{u}\right)\left(\frac{Cq^2}{u}\right) - \left(\frac{Cs}{u}\right)\left(\frac{Cs}{u} + mu\right)} \qquad (2)$$

Multiplying out the denominator gives

$$\frac{1}{u^2}\left[C_fC_r l^2 - Csmu^2\right]$$

So
$$\frac{\Omega_{ss}}{u} = \frac{-Cs(Y + C_f\delta) + C(xY + aC_f\delta)}{C_fC_r l^2 - Csmu^2} \qquad (3)$$

For $\Omega_{ss} = 0$ & $u < u_c$: $\cancel{C}Y(x-s) + \cancel{C}C_f\delta(a-s) = 0$

$$\Rightarrow \delta = -\frac{(x-s)Y}{C_f(a-s)} \qquad (4)$$

[If $a = s$: $\dfrac{aC_f - bC_r}{C_f + C_r} = a \Rightarrow a\cancel{C_f} - bC_r = a\cancel{C_f} + aC_r$

$\Rightarrow (a+b)C_r = 0 \Rightarrow C_r = 0$

ie vehicle has only one wheel and therefore spins under the action of the yaw moment]

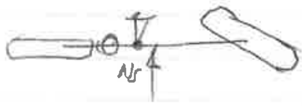
1 Cont

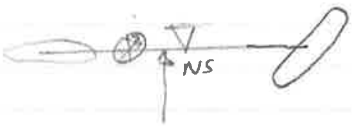
$$\delta = \frac{-(x-s)Y}{C_f(a-s)} = \frac{(C_f+C_r)(s-x)}{C_f C_r \cdot L} \quad \text{for } \Omega_{ss}=0$$

for $s<0$ (understeer) $a-s>0$

for $s>0$ (oversteer) $|s|<a$ ie NS point is always within the frame $\Rightarrow$ so denominator is always +ve

If $x=s$, $\delta=0$ ie the force is applied at the neutral steer point

If $x>s$ $\delta<0$ ie turn into wind 

If $x<s$ $\delta>0$ ie turn away from wind 

(c) To find sideslip, substitute (4) into (1a), with $\Omega_{ss}=0$:

$$C/u\, v_{ss} + (C\,s/u + mu)\underbrace{\cancel{\Omega_{ss}}}_{0} = Y + C_f\,\delta_{ss}$$

$$v_{ss}/u = \beta_{ss} \Rightarrow C\beta_{ss} = Y + \frac{\cancel{C_f}(s-x)Y}{\cancel{C_f}(a-s)} = \frac{Y(a-s+s-x)}{a-s}$$

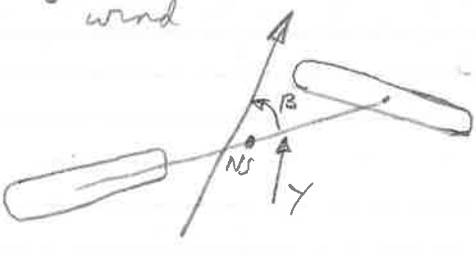
$$\text{So } \frac{C\beta_{ss}}{Y} = \frac{a-x}{a-s}$$

x can't be $>a$ because the force would be applied forward of the front wheel. So numerator is always >0

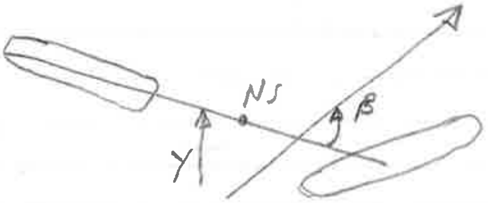
Denominator is always >0 because $|s|<a$

Conclusion: $\frac{C\beta_{ss}}{Y} > 0$ for all conditions ie the vehicle always has a steady sideslip velocity away from the wind

$x>s$

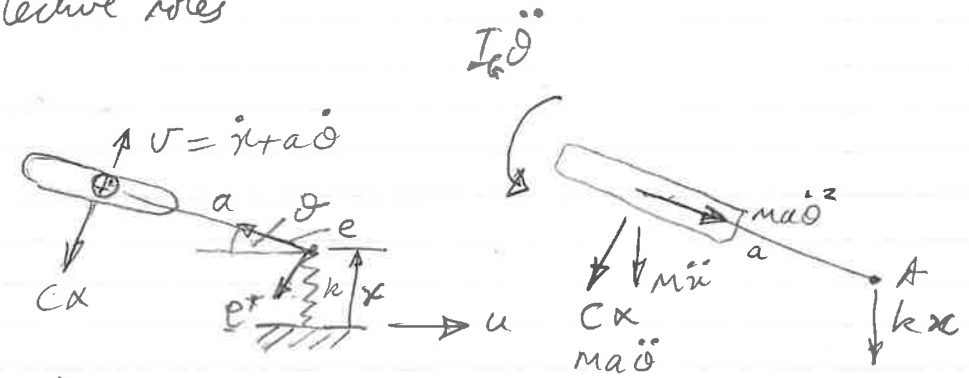


$x<s$



2a See lecture notes

2b.



$$\underline{R}_G = x\underline{j} + a\underline{e}$$

$$\dot{\underline{R}}_G = \dot{x}\underline{j} + a\dot{\theta}e^*$$

$$\ddot{\underline{R}}_G = \ddot{x}\underline{j} + a\ddot{\theta}\underline{e}^* - a\dot{\theta}^2\underline{e}$$

<u>Slip angle</u> $\alpha \simeq \dfrac{\dot{x} + a\dot{\theta}}{u} + \theta$

$$\downarrow \sum F_V: \quad m(\ddot{x} + a\ddot{\theta}) + c\left(\frac{\dot{x} + a\dot{\theta}}{u} + \theta\right) + kx = 0$$

$$\circlearrowleft \sum M_A: \quad I_G\ddot{\theta} + a\left[m(\ddot{x} + a\ddot{\theta}) + c\left(\frac{\dot{x} + a\dot{\theta}}{u} + \theta\right)\right] = 0$$

in matrix form

$$\begin{bmatrix} m & ma \\ ma & I_G + ma^2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c/u & ca/u \\ ca/u & ca^2/u \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k & c \\ 0 & ca \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = 0$$

Let $\underset{\sim}{q} = \begin{Bmatrix} X \\ \Theta \end{Bmatrix} e^{st}$ Then the characteristic eqn is:

$$\begin{vmatrix} ms^2 + cs/u + k & mas^2 + \frac{ca}{u}s + c \\ mas^2 + \frac{cas}{u} & (I_G + ma^2)s^2 + \frac{ca^2s}{u} + ca \end{vmatrix} = \underline{0}$$

from which:

$$0 = (ms^2 + cs/u + k)\left[(I_G + ma^2)s^2 + \frac{ca^2s}{u} + ca\right] - \left(mas^2 + \frac{cas}{u} + c\right)\left(mas^2 + \frac{cas}{u}\right)$$

$$\Rightarrow s^4\left[m(I_G + \cancel{ma^2}) - \cancel{m^2a^2}\right] + s^3\left[\cancel{\frac{mca^2}{u}} + \frac{c}{u}(I_G + \cancel{ma^2}) - \cancel{\frac{mca^2}{u}} - \frac{mca^2}{u}\right]$$

$$+ s^2\left[k(I_G + ma^2) + \cancel{\frac{c^2a}{u^2}} + \cancel{mca} - \cancel{mac} - \cancel{\frac{c^2a^2}{u^2}}\right]$$

2cont

$$+s\left[\cancel{\frac{c^2a}{u}} + \frac{ca^2k}{u} - \cancel{\frac{c^2a}{u}}\right] + kca$$

ie $$s^4\underbrace{(mI_G)}_{a_4} + s^3\underbrace{\left(\frac{I_G c}{u}\right)}_{a_3} + s^2\underbrace{k(I_G + ma^2)}_{a_2}$$
$$+ s\underbrace{\left(\frac{ca^2k}{u}\right)}_{a_1} + \underbrace{kca}_{a_0}$$

Stability conditions (Routh - Hurwitz)

(i) All a's +ve $\Rightarrow u > 0$

(ii) $a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2$

$$\Rightarrow k\left(\frac{\cancel{c}a^2k}{\cancel{u}}\right)(I_G + ma^2)\left(\frac{I_G \cancel{c}}{\cancel{u}}\right) > (kca)\left(\frac{I_G \cancel{c}}{\cancel{u}}\right)^2 + (mI_G)\left(\frac{\cancel{c}a^2k}{\cancel{u}}\right)^2$$

$$\Rightarrow k\cancel{I_G}a^{\cancel{2}}\cancel{k}(I_G + ma^2) > I_G^{\cancel{2}}\cancel{k}c\cancel{a} + m\cancel{I_G}a^{\cancel{4}\,3}k^{\cancel{2}}$$

$$ka(\cancel{I_G} + \cancel{ma^2}) > \cancel{I_G}c + \cancel{ma^3k}$$ Dimensionally correct ✓

$$\Rightarrow \underline{\underline{k > c/a}}$$

3(a)

Vertical displacement spectrum is of the form:

$$S_z(n) = \kappa\, n^{-w}$$

where $S_z(n)$ is single-sided MSSD (m^3/cycle)
n is wavenumber (cycle/m)
w is slope
κ is roughness (in compatible units)

In fig 1(a) it is evident that $w=2$ (since S_z drops two decades for one decade of n)
Hence differentiating the profile to give a velocity profile will scale the spectral density by n^2 giving a zero slope, hence can be idealised as white noise velocity.

(b) (i) Mean square dynamic tyre force should be minimised in order to maximise the cornering and braking forces achievable by the tyres, and to minimise damage to infrastructure (roads & bridges)

$$\frac{dE(\,)}{dk} = \frac{\pi S_0\left(2k(m_u+m_s)^3 - 2(m_u+m_s)m_u m_s k_t\right)}{m_s^2 c}$$

equate to zero $\cancel{2}k(m_u+m_s)^{\cancel{3}\,2} = \cancel{2}(\cancel{m_u+m_s})m_u m_s k_t$

$$\underline{\underline{k = \frac{m_u m_s}{(m_u+m_s)^2}\, k_t}}$$

If $m_u \ll m_s$ $\qquad \underline{\underline{k \simeq \frac{m_u}{m_s} k_t}}$

3b (ii) let $E(\ ) = \dfrac{\pi S_o \left(A + (m_u + m_s)^2 k_t c^2\right)}{m_s^2 c}$

where $A = (m_u+m_s)^3 k^2 - 2(m_u+m_s) m_u m_s k_t k + m_u m_s^2 k_t^2$

$$\frac{dE(\ )}{dc} = -\frac{\pi S_o A}{m_s^2 c^2} + \frac{\pi S_o (m_u+m_s)^2 k_t}{m_s^2}$$

equate to zero $\dfrac{\pi S_o A}{m_s^2 c^2} = \dfrac{\pi S_o (m_u+m_s)^2 k_t}{m_s^2}$

$$c^2 = \frac{A}{(m_u+m_s)^2 k_t}$$

$$c = \sqrt{\frac{(m_u+m_s)^3 k^2 - 2(m_u+m_s) m_u m_s k_t k + m_u m_s^2 k_t^2}{(m_u+m_s)^2 k_t}}$$

now $m_u \ll m_s$ and $k \sim \dfrac{m_u}{m_s} k_t$

so $m_u + m_s \to m_s$ and $\dfrac{m_u}{m_s} \ll 1$

thus

$$c = \sqrt{\frac{\frac{m_u^2}{m_s^2} m_s^3 k_t^2 - 2 m_s^2 m_u k_t \frac{m_u}{m_s} k_t + m_u m_s^2 k_t^2}{m_s^2 k_t}}$$

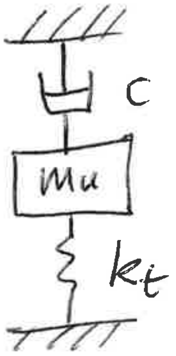
$$c = \sqrt{\frac{m_u^2}{m_s} k_t - 2\frac{m_u^2}{m_s} k_t + m_u k_t}$$

$$= \sqrt{m_u k_t \left(\frac{m_u}{m_s} - \frac{2 m_u}{m_s} + 1\right)}$$

$$\underline{c \simeq \sqrt{m_u k_t}}$$

iii) One could postulate that setting c to minimise dynamic tyre force involves achieving a desirable damping ratio for each of the two modes of vibration.

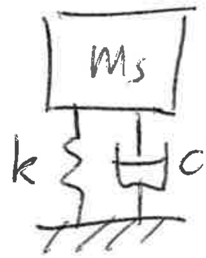
The unsprung mass mode can be idealised as:



From the mechanics data book, the damping ratio $\zeta_u = \sqrt{\frac{c}{2\sqrt{k_t m_u}}}$

Thus for given ζ_u, $\underline{\underline{c \propto \sqrt{k_t m_u}}}$

The spring mass mode can be idealised as:
(assuming $k << k_t$)



damping ratio $\zeta_s = \sqrt{\frac{c}{2\sqrt{k m_s}}}$

but from (b)(i) $k = \frac{m_u}{m_s} k_t$

So $\zeta_s = \sqrt{\frac{c}{2\sqrt{k_t m_u}}}$

and for given ζ_s, $\underline{\underline{c \propto \sqrt{k_t m_u}}}$

Thus, providing k is set according to (b)(i), the damping ratios of both modes are not determined by m_s.

4 (a)

- model has only 2 dof, real vehicle has 6 dof sprung mass, also unsprung masses, engine mass, flexural modes of body structure.
- real vehicle is excited in bounce and roll, due to travel along two parallel tracks, compared to single track of model.
- model is linear, real vehicle is nonlinear due to kinematics of suspension, friction, nonlinear spring and damper characteristics.

(b) Model is symmetrical about centre of mass, so modes of vibration are pure bounce and pure pitch about centre of mass.

pure bounce nat freq $\sqrt{\frac{2k}{m}} = \sqrt{\frac{2.80.10^3}{1000}}$

$= 12.65$ rad/s

$= \underline{\underline{2.01}}$ Hz.

pure pitch nat freq $\sqrt{\frac{2ka^2}{I}} = \sqrt{\frac{2.80.10^3.1.2^2}{1500}}$

$= 12.39$ rad/s

$= \underline{\underline{1.97}}$ Hz.

(c) time delay between axles is $\frac{2a}{u}$

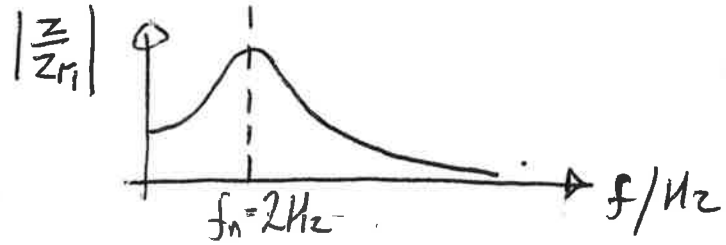
hence $Zr_2(j\omega) = e^{-j\omega\frac{2a}{u}} Zr_1(j\omega)$

thus $\begin{Bmatrix} z \\ \theta \end{Bmatrix} = \begin{bmatrix} H_z & H_z \\ -H_\theta & H_\theta \end{bmatrix} \begin{Bmatrix} 1 \\ e^{-j\omega\frac{2a}{u}} \end{Bmatrix} Zr_1$

$$\therefore \frac{z}{Zr_1} = H_z\left(1 + e^{-j\omega\frac{2a}{u}}\right), \quad \frac{\theta}{Zr_1} = H_\theta\left(-1 + e^{-j\omega\frac{2a}{u}}\right)$$

4 cont

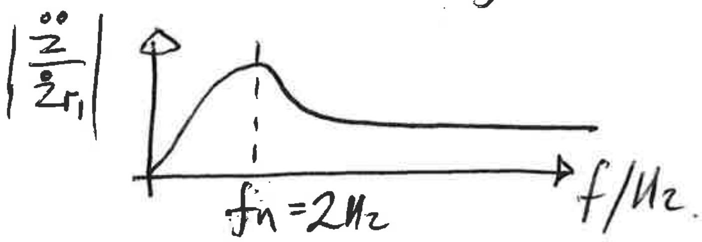
(d) start from transfer function of 1 dof system case(c) in mechanics data book.



ignoring wheelbase filtering

f is frequency in Hz

scale by frequency to account for vel in, accn out:



now account for wheelbase filtering

zeros in $\frac{z}{z_{r_1}}$ when $e^{-\frac{j\omega 2a}{u}} = -1 = e^{-j(\pi + n2\pi)}$ n is integer

$$\therefore \frac{\omega 2a}{u} = \pi + n2\pi$$

$\omega = 2\pi f$

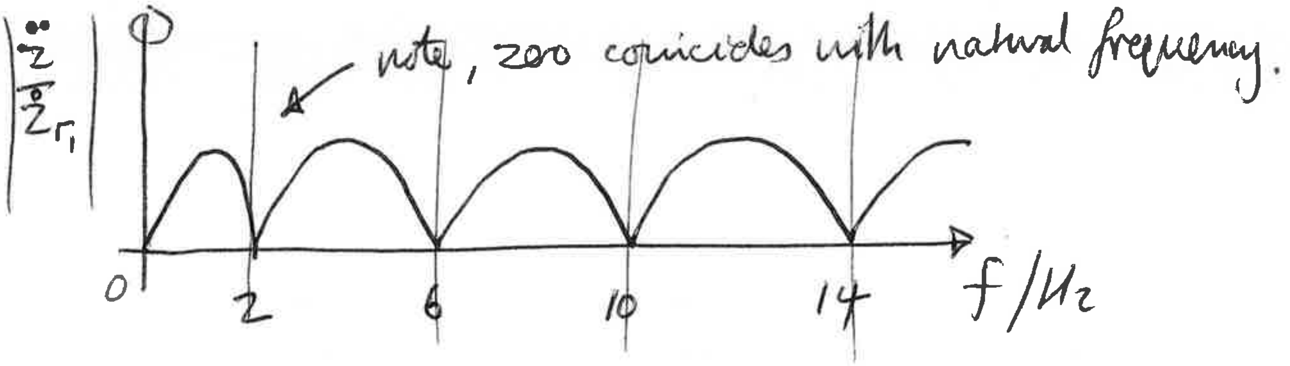
$$\frac{\cancel{2\pi} f\, 2a}{u} = \cancel{\pi} + n2\cancel{\pi}$$

$$\frac{4a}{u} f = 1 + 2n$$

$$f = (1+2n)\frac{u}{4a} = (1+2n)\frac{9.6}{4.1\cdot 2}$$

$$= (1+2n)\,2$$

$$f = 2, 6, 10, 14, 18 \ldots Hz.$$



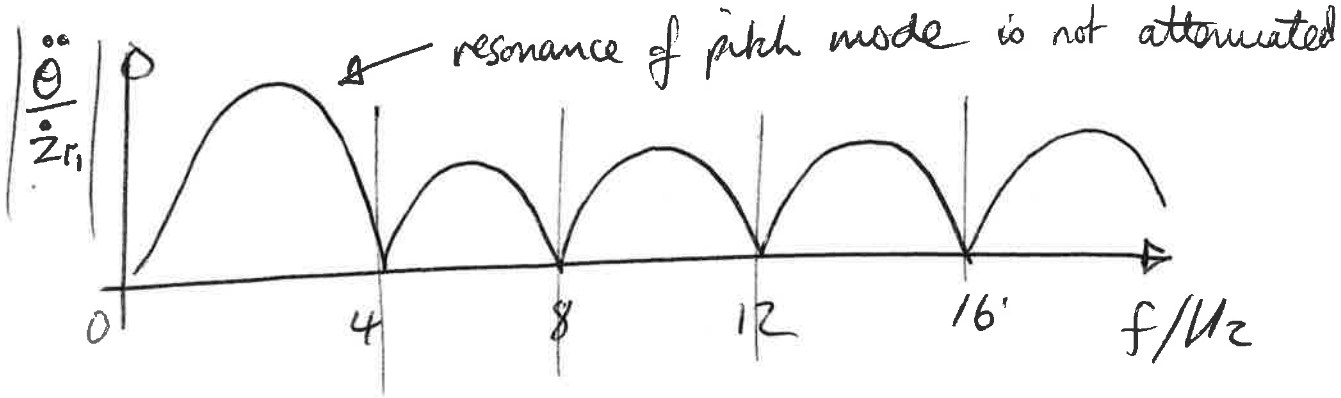
zeros in $\frac{\theta}{z_{r_1}}$ when $e^{-j\omega\frac{2a}{u}} = 1 = e^{-jn2\pi}$

$$\frac{\omega 2a}{u} = n2\pi$$

$$\omega = 2\pi f \qquad \frac{2\pi f 2a}{u} = n2\pi$$

$$f = \frac{un}{2a} = \frac{n \cdot 9.6}{2 \cdot 1.2} = 4n$$

$$f = 0, 4, 8, 12, 16 \ldots \text{Hz}$$



ENGINEERING TRIPOS PART IIB 2013

MODULE *4C8 Applications of Dynamics*

Assessor's Comments

Q1 Car subject to side wind

Very popular question - attempted by all but two candidates. Part (a) was bookwork and well done. Parts (b) and (c) were generally fine, though some students had little idea of how to proceed.

Q2 Wheel Shimmy

Part (a) was done well. The candidates were surprisingly weak at deriving the two equations of motion – particularly taking moments about the towing point. This is second year work…disappointing.

Q3 Quarter Car Dynamic Tyre Forces

Part (a) was generally fine. All candidates succeeded with (b)(i); there were some good solutions to (b)(ii), but there were no convincing answers to part (b)(iii).

Q4 Pitch-plane Model

An unpopular question, but generally well done by those who attempted it. This material is generally understood well by the students.