Crib 4C9 (2013)

1 (a)

(i) $\delta_{ij}\delta_{ik}\delta_{jk} = \delta_{1j}\delta_{1k}\delta_{jk} + \delta_{2j}\delta_{2k}\delta_{jk} + \delta_{3j}\delta_{3k}\delta_{kj}$

$= 3$ (10%)

(ii) $\delta_{ij}\delta_{ij} = \delta_{1j}\delta_{1j} + \delta_{2j}\delta_{2j} + \delta_{3j}\delta_{3j} = 3$ (10%)

(iii) $\delta_{ij}\delta_{jk} = \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k}$ (10%)

$= \delta_{ik}$

(b)

(i) Let $D_{ij} = D_{ij}^{u} + D_{ij}^{s}$ where D_{ij}^{u} is in asymmetric & D_{ij}^{s} the symmetric parts

$$D_{ij} = \frac{1}{2}(D_{ij} + D_{ji}) + \frac{1}{2}(D_{ij} - D_{ji}) = D_{ij}^{s} + D_{ij}^{u}$$

Then

$$D_{ij}^{s} x_i x_j = \frac{1}{2}(D_{ij} + D_{ji})x_i x_j = \frac{1}{2}(D_{ij} x_i x_j + D_{pq} x_q x_p)$$

$$= D_{ij} x_i x_j$$ (25%)

(ii) $D_{ij} x_i x_j = D_{1j} x_1 x_j + D_{2j} x_2 x_j + D_{3j} x_3 x_j$ (30%)

$$= D_{11}\cancel{x_1 x_1}^{0} + D_{12}x_1x_2 + D_{13}x_1x_3 + D_{21}x_2x_1 + D_{22}\cancel{x_2 x_2}^{0}$$
$$+ D_{23}x_2x_3 + D_{31}x_3x_1 + D_{32}x_3x_2 + D_{33}\cancel{x_3 x_3}^{0}$$

$D_{11} = D_{22} = D_{33} = 0$ & $D_{12} = -D_{21}$ etc $\Rightarrow$ above expression $= 0$

(c)

$$\varepsilon_{ij} = \cancel{\gamma_{ij}}\ \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

& the principal strain $\varepsilon^* = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(2)

(a) $\phi = Ar^2 \theta^m$

$$\frac{\partial \phi}{\partial r} = 2Ar\theta^m \; ; \quad \frac{\partial^2 \phi}{\partial r^2} = 2A\theta \; ; \quad \frac{\partial^2 \phi}{\partial \theta^2} = m(m-1)Ar^2\theta^{m-2}$$

$$\nabla^2 \phi = \left[4\theta^2 + m(m-1)\right] A\theta^{m-2}$$

$$\nabla^4 \phi = \left[4m(m-1)\theta^2 + m(m-1)(m-2)(m-3)\right] A\theta^{m-4} = 0$$

$$\Rightarrow \; m = 0, 1 \;\; \& \;\; \cancel{m = \frac{5}{2} \pm \frac{1}{2}\sqrt{1+16\theta^2}}$$

(fcn of θ)

(20%)

(b) $\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 2A\theta, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = 2A\theta$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -A$$

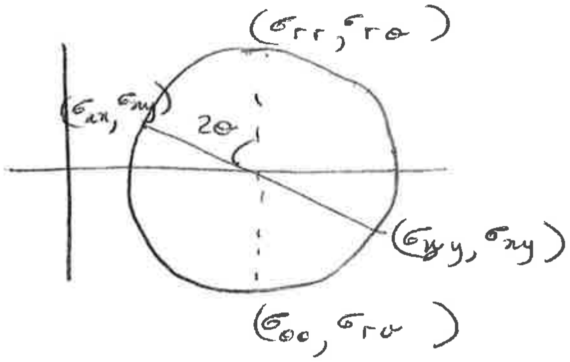
tractions on surface of half-space

$\sigma_{\theta\theta}(r, 0) = 0, \qquad \sigma_{r\theta}(r, 0) = -A$

$\sigma_{\theta\theta}(r, \pi) = 2\pi a \qquad \sigma_{r\theta}(r, \pi) = -A$

(40%)

Stress transformation $(r, \theta) \rightarrow x, y$



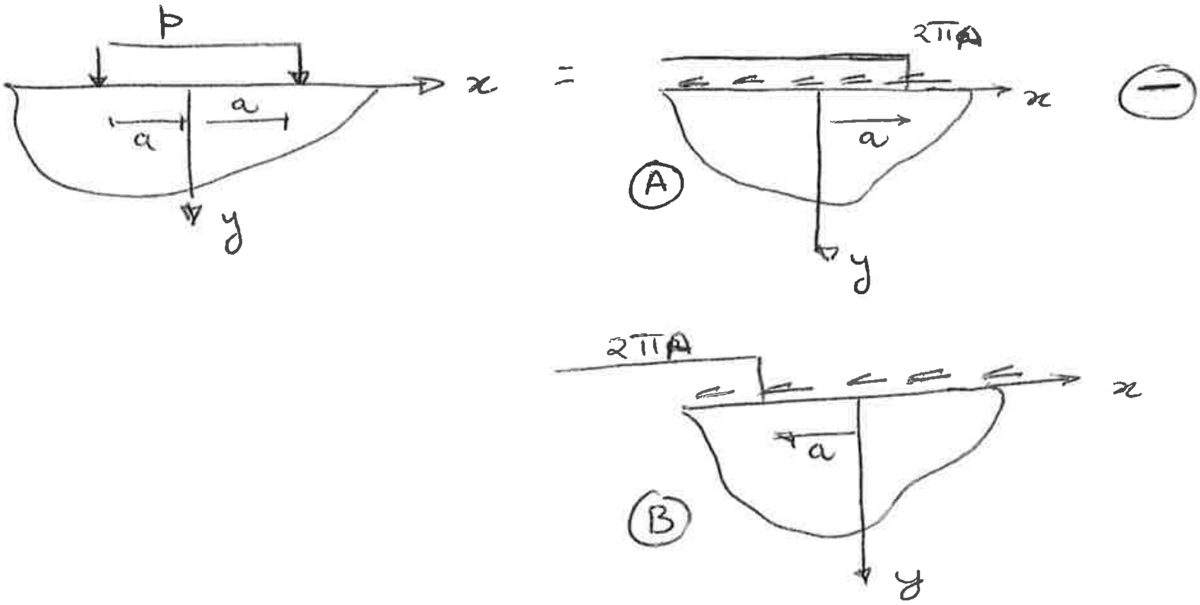
$$\sigma_{xx} = A(2\theta - \sin 2\theta)$$
$$= A\left[2\tan^{-1}\frac{y}{x} - \frac{2xy}{x^2+y^2}\right]$$

$$\sigma_{yy} = A(2\theta + \sin 2\theta)$$
$$= A\left[2\tan^{-1}\frac{y}{x} + \frac{2xy}{x^2+y^2}\right]$$

$$\sigma_{xy} = -A\cos 2\theta = -A\left(\frac{x^2-y^2}{x^2+y^2}\right)$$

(c) for $\sigma_{yy}(x,0) = -p$ in $-a < x < a$ & $\sigma_{yy}(x,0) = 0$ for $|x| > a$

& $\sigma_{xy}(x,0) = 0$ for all x.



$$(A) \quad \sigma_{xy} = -A \frac{(x-a)^2 - y^2}{(x-a)^2 + y^2}$$

$$(B) \quad \sigma_{xy} = -A \frac{(x+a)^2 - y^2}{(x+a)^2 + y^2}$$

$$\sigma_{yy}(x,0) = -p = 2\pi A \Rightarrow A = -\frac{p}{2\pi}$$

shear stress $\sigma_{xy}(x,y) = \frac{p}{2\pi}\left[\frac{(x-a)^2 - y^2}{(x-a)^2 + y^2} - \frac{(x+a)^2 - y^2}{(x+a)^2 + y^2}\right]$

$$\sigma_{xy}(x,a) = \frac{p}{2\pi}\left[\frac{(x-a)^2 - a^2}{(x-a)^2 + a^2} - \frac{(x+a)^2 - a^2}{(x+a)^2 + a^2}\right]$$

(40%)

$$= -\frac{p}{2\pi} \frac{4a^3 x}{(x^2+2a^2)^2 - 4a^2x^2}$$

3 (a) Druckers postulates are only valid for stable materials; energy cannot be extracted from such materials in a closed cycle of applied stress.

From Druckers postulates we can show that ~~the plastic strain rate is~~ (i) normality & (ii) convexity. (20%)

(b) $J_2 = \frac{1}{2} s_{ij} s_{ij}$; $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

$$\dot{J}_2 = \frac{\partial J_2}{\partial \sigma_{ij}} \dot{\sigma}_{ij}$$

but $$\frac{\partial J_2}{\partial \sigma_{ij}} = s_{pq} \frac{\partial s_{pq}}{\partial \sigma_{ij}} = s_{pq}\left[\delta_{ip}\delta_{jq} - \frac{1}{3}\delta_{ij}\delta_{pq}\right]$$

$$= s_{pq}\delta_{ip}\delta_{jq} - \frac{1}{3} s_{pp}\delta_{ij}$$

$$= s_{ij} = \frac{\partial J_2}{\partial s_{ij}}$$

$$\Rightarrow \dot{J}_2 = s_{ij}\dot{\sigma}_{ij}$$ (25%)

(c) $$\sigma_1 = \frac{\sigma_{11}}{2} + \left(\frac{\sigma_{11}^2}{4} + \sigma_{12}^2\right)^{1/2}$$

$$\sigma_2 = \frac{\sigma_{11}}{2} - \left(\frac{\sigma_{11}^2}{4} + \sigma_{12}^2\right)^{1/2}$$

$$\sigma_3 = 0$$

$$\Rightarrow J_2 = \frac{1}{3}\left(\sigma_{11}^2 + 3\sigma_{12}^2\right)$$

$$\sigma_e = \sqrt{3J_2} = \sqrt{\sigma_{11}^2 + 3\sigma_{12}^2}$$

$$\dot{J}_2 = \frac{2}{3}\left(\sigma_{11}\dot{\sigma}_{11} + 3\sigma_{12}\dot{\sigma}_{12}\right)$$

From J_2 - flow theory

$$\dot{\varepsilon}_{ij}^{\,p} = \frac{1}{h} s_{ij} \dot{J}_2$$

where $\frac{1}{h} = \frac{9}{4\sigma_e^2}\left(\frac{1}{E_T} - \frac{1}{E}\right)$

$$\Rightarrow \dot{\varepsilon}_{11} = \frac{\dot{\sigma}_{11}}{E} + \frac{1}{h} s_{11} \frac{2}{3}\left(\sigma_{11}\dot{\sigma}_{11} + 3\sigma_{12}\dot{\sigma}_{12}\right)$$

$$\& \; s_{11} = \frac{2}{3}\sigma_{11}$$

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} + \int_0^{\sigma_{11}} \frac{\left(\frac{2\sigma_{11}}{3}\right)^2 d\sigma_{11}}{\sigma_{11}^2 + 3\left(\frac{\sigma_T}{\sqrt{3}}\right)^2} \frac{9}{4}\left(\frac{1}{E_T} - \frac{1}{E}\right)$$

$$= \frac{\sigma_{11}}{E} + \left(\frac{1}{E_T} - \frac{1}{E}\right)\int_0^{\sigma_{11}} \frac{d\sigma_{11}}{1 + \left(\frac{\sigma_T}{\sigma_{11}}\right)^2}$$

$$= \frac{\sigma_{11}}{E} + \left[\frac{1}{E_T} - \frac{1}{E}\right]\left[\sigma_{11} - \sigma_T \tan^{-1}\left(\frac{\sigma_{11}}{\sigma_T}\right)\right]$$

(50%)