1(a) Essentially bookwork

(a) Hertz - Smooth surfaces, linear elasticity - small strains - no friction...
At the small scale, van der Waal forces both in and (perhaps) in the vicinity of the contact spot can influence the size of the contact and the geometry of the deformation. At larger scales these effects are usually negligible both because the range of surface forces is short and because of roughness effects.

(b) (i) JKR from Data sheet

$$P = \frac{4E^* a^3}{3R} - 2\sqrt{2\pi w E^*}\, a^{3/2}$$

first term is Hertz and second effect of adhesion within the contact.

BCP $$P = \frac{4E^* a^3}{3R} - \sqrt{2\pi w E^*}\, a^{3/2} - \pi w R$$

last term on RH side is independent of contact spot size a and thus must represent effect of attractive forces outside the contact.

(ii) from JKR above if $P=0$ then

$$\frac{4E^*}{3R} a^3 = 2\sqrt{2\pi w E^*}\, a^{3/2}$$

i.e. $a=0$ or $a = \left(\frac{9\pi w R^2}{2E^*}\right)^{1/3}$ as given

(as over) Solving BCP for $P=0$ means solving quadratic in $a^{3/2}$. Easier to substitute in given value of a

then $$P = \frac{4E^*}{3R} \cdot \frac{9\pi w R^2}{E^*} - \sqrt{2\pi w E^*}\sqrt{\frac{9\pi w R}{E^*}} - \pi w R$$

$$= 6\pi w R - 3\pi w R - \pi w R = 2\pi w R$$

So to make $P=0$, a must be decreased

$$\frac{4E^*}{3R}a^3 - \sqrt{2\pi w E^*}\, a^{3/2} - \pi w R = 0$$

$$a^{3/2} = \frac{\sqrt{2\pi w E^*} \pm \left[2\pi w E^* + 4\pi w R \,.\, 4E^*/3R\right]^{1/2}}{8E^*/3R}$$

$$\therefore \quad \frac{8E^*}{3R}.a^{3/2} = \sqrt{2\pi w E^*} \pm \left[2\pi w E^* + 16\pi w E^*/3\right]^{1/2}$$

$$\frac{8E^*}{3R}\, a^{3/2} = \left\{\sqrt{2} + \sqrt{\frac{22}{3}}\right\}\left[\pi w E^*\right]^{1/2}$$

$$a^3 = \left\{\sqrt{2} + \sqrt{22/3}\right\}^2 \frac{\pi w E^* \,.\, 9R^2}{8.8E^{*2}}$$

$$\therefore \quad a = \left\{\sqrt{2} + \sqrt{22/3}\right\}^{2/3}\left(\frac{9\pi}{64}\right)^{1/3}\left(\frac{wR^2}{E^*}\right)^{1/3}$$

$$a = 1.95\left(\frac{wR^2}{E^*}\right)^{1/3}$$

JKR $\quad a = \left(\frac{9\pi}{2}\right)^{1/3}\left(\frac{wR^2}{E^*}\right)^{1/3} = 2.42\left(\frac{wR^2}{E^*}\right)^{1/3}$

(iii) When $\frac{dP}{da} = 0$, P will be $-$ve ie. a tension, so

now contact is unstable and this represents pull-off.

(iv) Differentiating JKR $\quad \frac{dP}{da} = \frac{4E^*}{R}a^2 - 3\sqrt{2\pi w E^*}\, a^{1/2}$

when $\frac{dP}{da} = 0 \quad a^3_{JKR} = \frac{9\pi w R^2}{8E^*}$

Substituting gives pull-off force $\quad P_{JKR} = -\frac{3\pi w R}{2}$

Similarly for BCP

$$\frac{dP}{da} = \frac{4E^* a^2}{R} - \frac{3}{2}\sqrt{2\pi w E^*}\, a^{1/2}$$

$$\therefore \quad a^3_{BCP} = \frac{9\pi}{32}\frac{wR^2}{E^*}$$

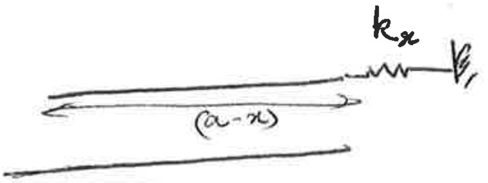
and substituting $\quad P_{BCP} = \frac{4}{3}\cdot\frac{9\pi}{32} wR - \left[2\pi w E^* \cdot \frac{9\pi}{32}\frac{wR^2}{E^*}\right]^{1/2} - \pi w$

$$= \frac{3\pi}{8} wR - \frac{3}{4}\pi wR - \pi wR$$

$$P_{BCP} = -\frac{11}{8}\pi wR$$

i.e. $\frac{11}{12} \cdot P_{JKR}$

Q2



(a)
$$W = \frac{1}{2} C V^2$$
$$= \frac{1}{2} \frac{t(a+x)\epsilon_0}{g} V^2$$

$$F_x = \frac{dW}{dx}$$
$$= \frac{\epsilon_0 t V^2}{2g} \quad \text{for a single actuator gap}$$
$$F_x = \frac{N \epsilon_0 t V^2}{2g} \quad \text{where } N = \text{no. of gaps}$$

(b) orthogonal to drive direction actuator appears as a parallel plate

$$F_y = \frac{N}{4} \frac{\epsilon_0 t (a+x) V^2}{(g-y)^2} - \frac{N}{4} \frac{\epsilon_0 t (a+x) V^2}{(g+y)^2}$$
$$= \frac{N}{4} \epsilon_0 t (a+x) V^2 \left[\frac{1}{(g-y)^2} - \frac{1}{(g+y)^2} \right]$$

(c) Stability criterion:

(Pull-in)
$$k_y = \left. \frac{\partial F_y}{\partial k_y} \right|_{y \to 0} = \frac{N}{4} \epsilon_0 t (a+x) V^2 \left[\frac{4}{g^3} \right]$$
$$x = \frac{F_x}{k_x} = \frac{N \epsilon_0 t V^2}{2 g k_x}$$
$$k_y = \frac{N \epsilon_0 t}{g^3} \left(a + \frac{N \epsilon_0 t V^2}{2 g k_x} \right) V^2$$
$$\therefore V^2 = V_{SI}^2 = \frac{ -\frac{N \epsilon_0 t a}{g^3} + \sqrt{ \left(\frac{N \epsilon_0 t a}{g^3} \right)^2 + 4 \frac{N \epsilon_0 t k_y}{2 g k_x} \left(\frac{N \epsilon_0 t}{g^3} \right) } }{ 2 \frac{N \epsilon_0 t}{g^3} \left(\frac{N \epsilon_0 t}{2 g k_x} \right) }$$

$$V_{SI}^2 = \frac{+\left[\frac{N\epsilon_0 t}{g^3}\right]\left(-a + \sqrt{a^2 + \frac{4N\epsilon_0 t k_y}{2gk_x}\left(\frac{g^3}{N\epsilon_0 t}\right)}\right)}{\left[\frac{N\epsilon_0 t}{g^3}\right] \qquad \left(\frac{N\epsilon_0 t}{g k_x}\right)}$$

$$V_{SI}^2 = \frac{\left(\sqrt{\frac{2k_y}{k_x}g^2 + a^2} - a\right) g k_x}{N\epsilon_0 t}$$

$$V_{SI}^2 = \frac{g^2 k_x}{N\epsilon_0 t}\left(\sqrt{\frac{2k_y}{k_x} + \left(\frac{a}{g}\right)^2} - \left(\frac{a}{g}\right)\right)$$

(d)

$$x_{SI} = \frac{N\epsilon_0 t}{2g}\,\frac{V_{SI}^2}{k_x}$$

$$x_{SI} = \frac{g}{2}\left(\sqrt{\frac{2k_y}{k_x} + \left(\frac{a}{g}\right)^2} - \left(\frac{a}{g}\right)\right)$$

Q3

(a) The equations can be written as:

$$m\ddot{x} + b_x\dot{x} + k_x x = f_{ext} \underbrace{\left(-2m\Omega\dot{y} - m\dot{\Omega}y + m\Omega^2 x\right)}_{\to 0}$$

$$m\ddot{y} + b_y\dot{y} + k_y y = -2m\Omega\dot{x} \underbrace{\left(-m\dot{\Omega}x + m\Omega^2 y\right)}_{\text{ignore at MEMS scale}}$$

$$x = A\cos\omega t \quad \text{(harmonic constant amplitude drive)}$$

$$\ddot{y} + \frac{\omega_y}{Q_y}\dot{y} + \omega_y^2 y = -2\cancel{m}\Omega\dot{x}$$

(b)

$$\ddot{y} + \frac{\omega_y}{Q_y}\dot{y} + \omega_y^2 y = +2\cancel{m}\Omega A\omega\sin\omega t$$

$$s^2Y(s) + \frac{\omega_y}{Q_y}sY(s) + \omega_y^2 Y(s) = -2\cancel{m}\Omega sX(s).$$

$$\therefore \quad Y(s) = \frac{-2\cancel{m}\Omega s\, X(s)}{s^2 + \omega_y^2 + \dfrac{s\omega_y}{Q_y}}$$

Putting $s = j\omega_x$

$$Y(j\omega) = \frac{2\cancel{m}\Omega j\omega_x X(j\omega)}{\omega_y^2 - \omega_x^2 + \dfrac{j\omega_x\omega_y}{Q_y}}$$

$$\text{or } |Y| = \frac{2\cancel{m}\Omega\omega_x A}{\sqrt{(\omega_y^2-\omega_x^2)^2 + \left(\dfrac{\omega_x\omega_y}{Q_y}\right)^2}}$$

mechanical sensitivity

$$\left|\frac{Y}{\Omega}\right| = \frac{2\cancel{m}\omega_x A}{\sqrt{(\omega_y^2-\omega_x^2)^2 + \left(\dfrac{\omega_x\omega_y}{Q_y}\right)^2}}$$

(c) If Ω is not constant but varying such that $\Omega = \Omega_0 \cos \omega_a t$ then the Coriolis force is:

$$F_c = 2m\Omega_0 A\omega_x \underbrace{\sin\omega_x t \cos\omega_a t}_{\left[\frac{\sin(\omega_x+\omega_a)t + \sin(\omega_x-\omega_a)t}{2}\right]}$$

Two forcing terms result for y and using the mechanical sensitivity expression:—

$$\left|\frac{Y}{\Omega_0}\right| \simeq \frac{\omega_x A}{\left|\omega_y^2-(\omega_x-\omega_a)^2+j\frac{(\omega_x-\omega_a)\omega_y}{Q_y}\right|} + \frac{\omega_x A}{\left|\omega_y^2-(\omega_x+\omega_a)^2+j\frac{(\omega_x+\omega_a)\omega_y}{Q_y}\right|}$$

For a MEMS gyroscope: $\omega_a << \omega_x, \omega_y$.
Also it is clear that for highest sensitivity $\omega_x = \omega_y$.

$$\left|\frac{Y}{\Omega_0}\right| \simeq \frac{2\omega_x A}{\sqrt{(4\omega_x\omega_a)^2+\left(\frac{\omega_x^2}{Q_y}\right)^2}}$$

The dependence between bandwidth and sensitivity is now established

if $\omega_a << \frac{\omega_x}{Q_y}$, scale factor approx. constant.

if $\omega_a \simeq \frac{\omega_x}{Q_y}$ or larger then, scale factor is a function of input frequency.

(d) For thermo-mechanical noise, equation Coriolis force to the noise f

$$(2m\Omega A\omega_x)^2 = 4k_BTb_y = \frac{4k_BT.m\omega_y}{Q_y}$$

$$\therefore \overline{\Omega_n^2} = \frac{k_BT\omega_y}{mA^2\omega_x^2Q_y}.$$

For improved signal-to-noise, reduce T, ↑ velocity in drive direction, ↑ m, ↑ Q_y.

Q4

(a) For a micro-resonator

$$\dot{X}(s) = \frac{sF(s)/m}{s^2 + \frac{s\omega_r}{Q_r} + \omega_r^2}$$

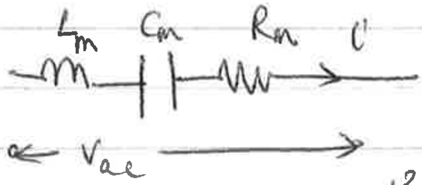
Note ‖l plate actuator $F(s) = \frac{\epsilon_0 A V_p}{g^2} v_{ac}(s)$.

comb-drive sense electrode $i = \frac{dCV_p}{dt} = V_p \frac{dC}{dt} = V_p \frac{dC}{dx}\frac{dx}{dt}$

$$i(s) = V_p \cdot \left(\frac{\epsilon_0 t}{g}\right) \dot{X}(s)$$

$$\therefore \frac{i(s)}{v_{ac}(s)} = \left(\frac{\epsilon_0 V_p}{g}\right)^2 \left(\frac{tA}{g}\right) \frac{s/m}{s^2 + \frac{s\omega_r}{Q_r} + \omega_r^2}$$

(b) Consider a series L-R-C equivalent circuit:



$$L_m \dot{i} + \frac{1}{C_m}\int i\,dt + R_m i = v_{ac}$$

$$s^2 L_m i + \frac{i}{C_m} + sR_m i = v_{ac}$$

$$\therefore i/v_{ac} = \frac{1}{s^2 L_m + sR_m + 1/C_m}$$

comparing the expression from (a) we get:

$$L_m = \frac{m}{\left(\frac{\epsilon_0 V_p}{g}\right)^2 \left(\frac{tA}{g}\right)}$$

$$C_m = \frac{m\omega_r^2}{\left(\frac{\epsilon_0 V_p}{g}\right)^2 \left(\frac{tA}{g}\right)}$$

(c) $$R_m = \frac{m\omega_r}{Q_r \left(\frac{\epsilon_0 V_p}{g}\right)^2 \left(\frac{tA}{g}\right)}.$$

To reduce R_m for a fixed ω_r, $\downarrow Q_r$, $\uparrow V_p$, $\underbrace{\downarrow g}_{\text{process}}$, $\underbrace{\uparrow t, \uparrow A}_{\text{topology/process}}$ and $\underbrace{\downarrow m}_{\text{mode shape engineeing}}$

(d) Nonlinearities inherent to the operation of resonator ultimately limit motional amplitude and hence output power. Physical origins include geometric nonlinearities (eg mechanical beam stiffening with amplitude), material nonlinearities (eg. material softening) and electrical nonlinearities (eg associate with parallel plate transduction).

(e) $$F_{\parallel\text{-plate}} = \frac{\epsilon_0 A}{(g-x)^2}\left(V_p^2 + v_{ac}V_p + v_{ac}^2\right) \quad ; x \text{ is displacement}$$

$$= \frac{\epsilon_0 A}{g^2\left(1-\frac{x}{g}\right)^2}\left(V_p^2 + v_{ac}V_p + v_{ac}^2\right)$$

$$\simeq \frac{\epsilon_0 A}{g^2}\left(1+\frac{2x}{g}\right)\left(V_p^2 + \ldots\right)$$

$$k_e = \frac{dF}{dx} \simeq \frac{2\epsilon_0 A V_p^2}{g^3}$$

$$\omega_r = \sqrt{\frac{k_m - k_e}{m}} = \sqrt{\frac{k_m - \frac{2\epsilon_0 A V_p^2}{g^3}}{m}}$$

$$Q_r = \frac{m\omega_r}{b_r}$$