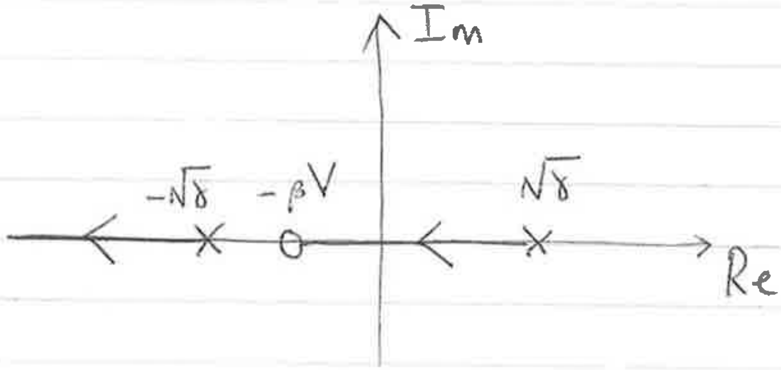
4F1 2013 solution

1 (a)(i)

$$\frac{s+\beta V}{(s-\sqrt{\gamma})(s+\sqrt{\gamma})}$$

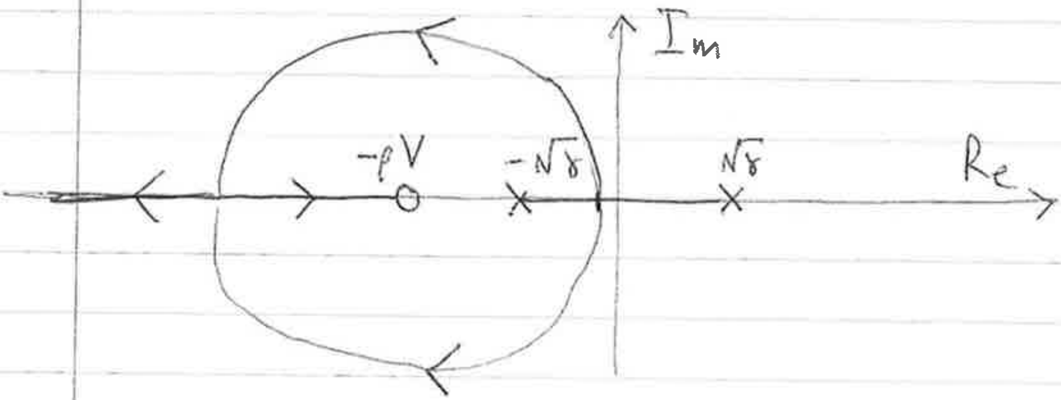
$\beta V < \sqrt{\gamma}$



(ii) Breakaway points: $\beta V > \sqrt{\gamma}$

$$\begin{aligned} 0 &= (s+\beta V)2s - (s^2-\gamma) \\ &= s^2 + 2\beta V s + \gamma \end{aligned}$$

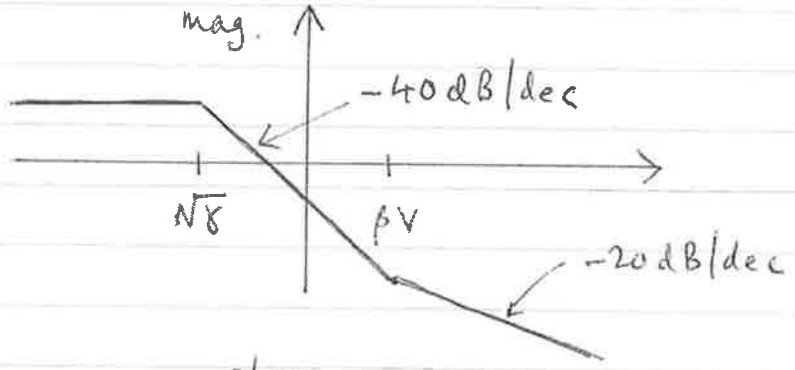
All coeffs. positive so roots in LHP by Routh test (data book

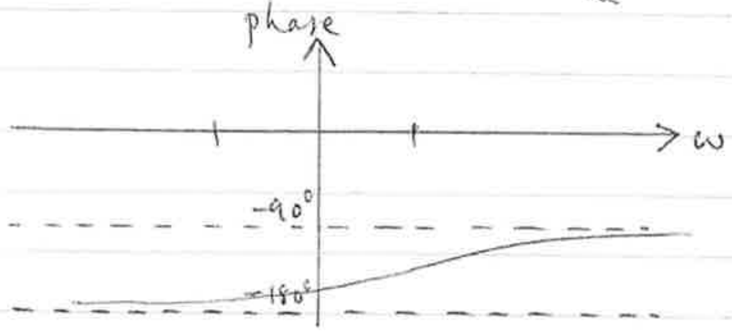


(b)

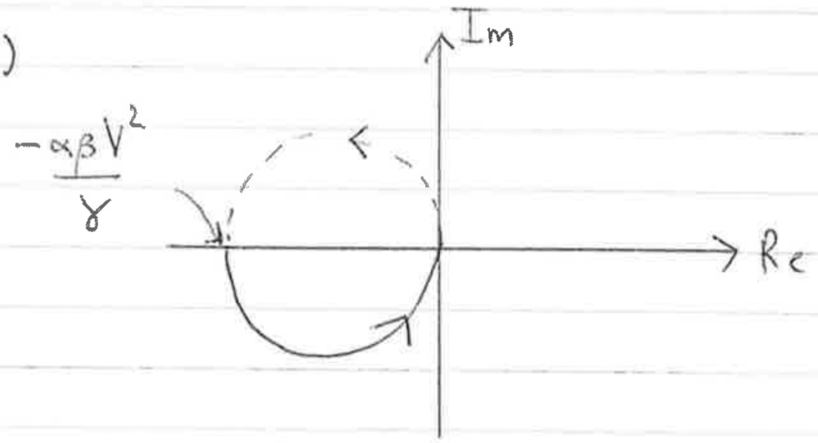
$$t.f. = \alpha V \frac{j\omega + \beta V}{-\omega^2 - \gamma}$$

poles contribute $-180°$ phase, constant





(c)



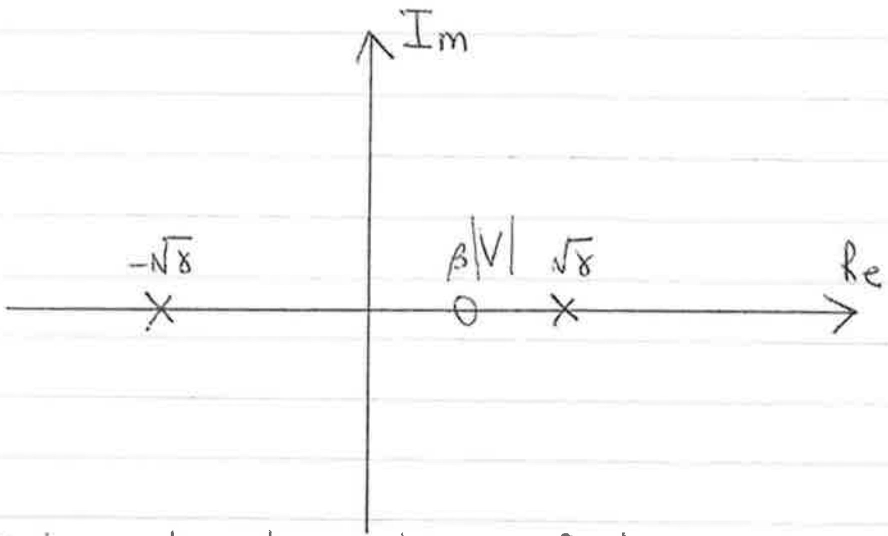
Need one counter-clockwise encirclement of $-\frac{1}{k}$ for closed-loop stability.

$$\Leftrightarrow \quad -\frac{1}{k} > -\frac{\alpha\beta V^2}{\gamma} \quad \Leftrightarrow \quad k > \frac{\gamma}{\alpha\beta V^2}$$

Otherwise there is one pole in RHP.

(d) System is open-loop unstable but is stabilised by proportional gain negative feedback - hence is relative easy to control. For higher forward speed less gain is needed

(e)

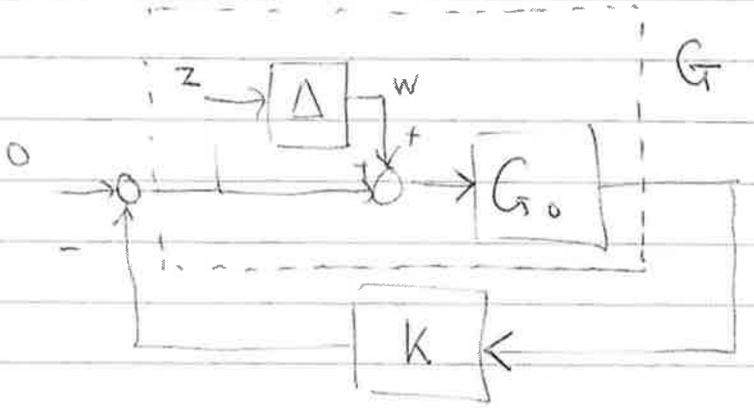


If $k(s)$ has all poles in the LHP there will always be a branch of root-locus diagram on the positive real axis for all k (positive or negative).

No longer true if $\beta|V| > \sqrt{\gamma}$. However, constant gain feedback cannot stabilise the plant.

It is probably no comfort to most riders that stabilisation is likely to be easier at high speeds!

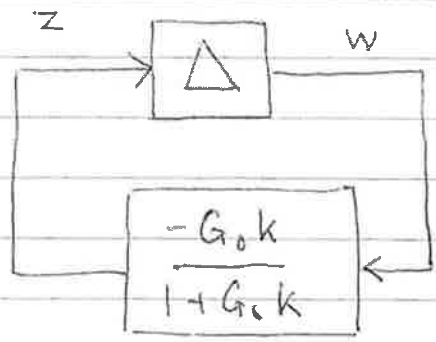
2(a)



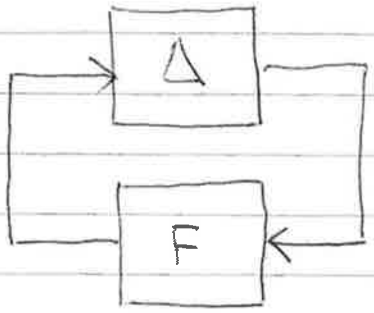
$$z = -G_0 K(w+z)$$

$$\Rightarrow z = \frac{-G_0 K}{1 + G_0 K}$$

↓ rearrange



Small gain:



Δ, F both stabi

Stable for all Δ
with $|\Delta(j\omega)| < h(\omega)$
if and only if

$$|F(j\omega)|^{-1} \leq h(\omega)^{-1}$$

for all ω.

Desired result follows.

2 (b) (i)

$$G(s) = \frac{as+b}{s^2+10s} = \frac{s+2}{s^2+10s}\,\frac{as+b}{s+2}$$

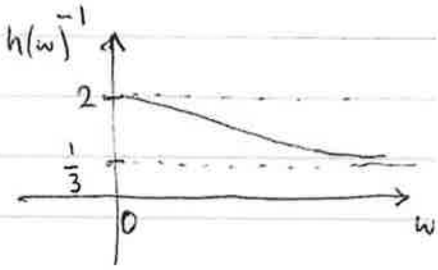
$$= \frac{s+2}{s^2+10s}\left(1 + \underbrace{\frac{(a-1)s+(b-2)}{s+2}}_{\Delta(s)}\right)$$

Note that $|a-1| < 3$ and $|b-2| < 1$. Hence

$$|\Delta(j\omega)| = \left|\frac{(a-1)j\omega + b-2}{j\omega + 2}\right| < \sqrt{\frac{9\omega^2+1}{\omega^2+4}}$$

(ii) $S(j\omega) + T(j\omega) = 1$

$$|T(j\omega)| \leq \sqrt{\frac{\omega^2+4}{9\omega^2+1}}$$



Impossible to achieve $|S(j\omega)| < \frac{1}{2}$ if $|T(j\omega)| < \frac{1}{2}$

$$\frac{\omega^2+4}{9\omega^2+1} = \frac{1}{4} \iff 4\omega^2+16 = 9\omega^2+1$$

$$\iff 5\omega^2 = 15 \iff \omega = \sqrt{3}$$

Impossible to achieve $|S(j\omega)| < \frac{1}{2}$ if $\omega > \sqrt{3}$ rad/sec.

(iii) (3) $\iff \left|\dfrac{k(j\omega+2)}{-\omega^2+(k+10)j\omega+2k}\right| \leq \sqrt{\dfrac{\omega^2+4}{9\omega^2+1}}$

$$\iff k^2(9\omega^2+1) \leq (-\omega^2+2k)^2 + (k+10)^2\omega^2$$

$$\iff \quad 0 \leq \omega^4 + \omega^2\left(-9k^2 - 4k + (k+10)^2\right) + 3k^2$$

$$= \omega^4 + \omega^2\left(-8k^2 + 16k + 100\right) + 3k^2$$

(iv)

$$1 + G\,K(s) = 1 + \frac{as+b}{s^2+10s}\,k = \frac{s^2+10s+k(as+b)}{s^2+10s}$$

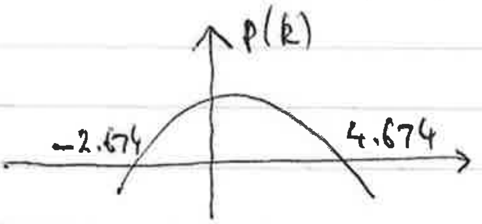
closed-loop stable $\iff$ all roots of $s^2 + (10+ka)s + kb$ in LHP

$$\iff \quad 10 + ka > 0 \quad \text{and} \quad kb > 0$$

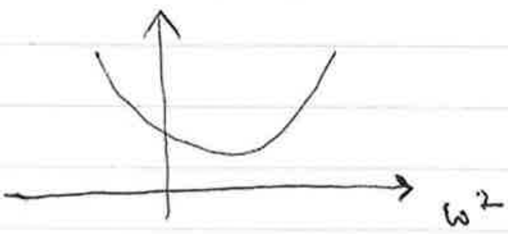
$$\iff \quad ak > -10 \quad \text{and} \quad k > 0$$

$$\iff \quad 0 < k < 5$$

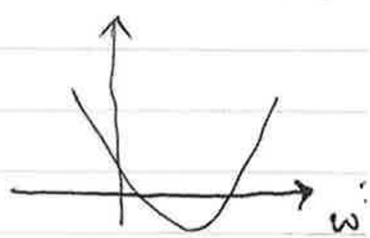
To check claim write: $p(k) = -8k^2 + 16k + 100$



Consider: $\omega^4 + \omega^2 p(k) + 3k^2$ (when $k > 4.674$)



or



Need discriminant negative: $-p(k) < 2\sqrt{3}\,k \iff k < 4.955$

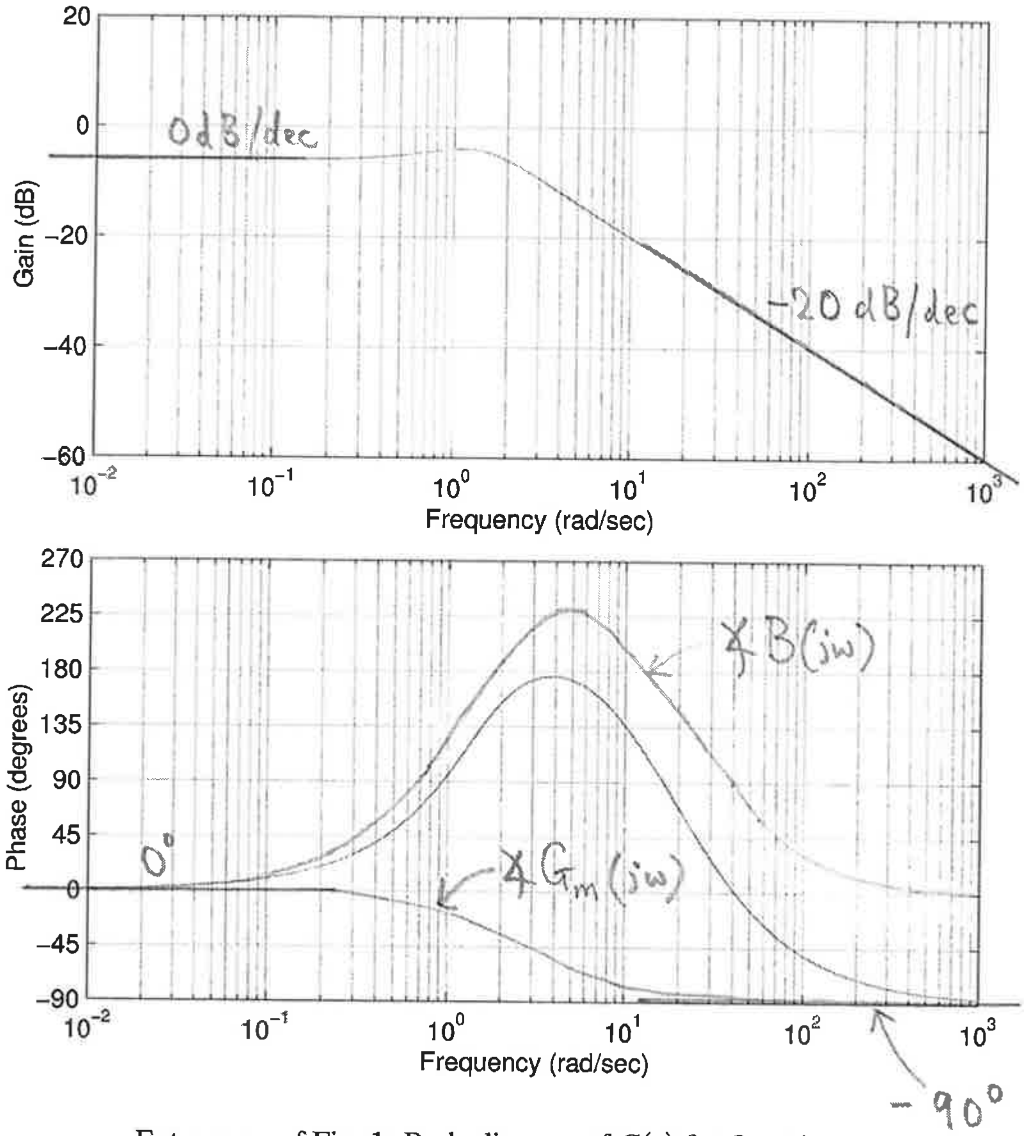
iv) cont.

The robustness result of part(a) is necessary and sufficient for general $\Delta(s)$. For a structured perturbation within the class the condition is no longer necessary, hence the true maximum range of k which can stabilise the class could be larger.

3 (a)

ENGINEERING TRIPOS PART IIB

Friday 13 May 2013, Module 4F1, Question 3.



Extra copy of Fig. 1: Bode diagram of $G(s)$ for Question 3.

(a) Accurate computer plot

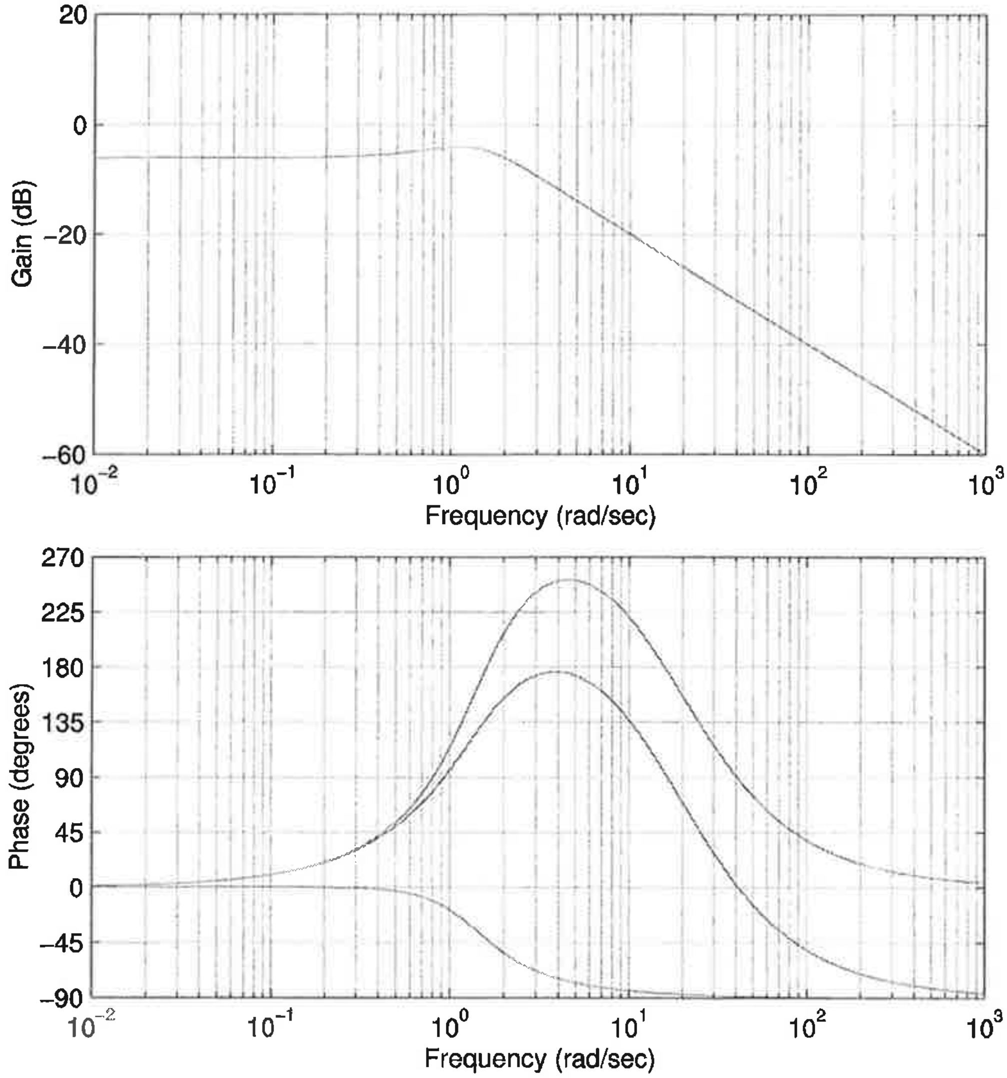


Fig. 1

(TURN OVER for continuation of Question 1

3 (b) From sketch:

pair of RHP poles around 1 rad/sec

" " " zeros " 20 rad/sec

$$\Rightarrow \quad B(s) = \left(\frac{s+1}{s-1}\right)^2 \left(\frac{s-20}{s+20}\right)^2$$

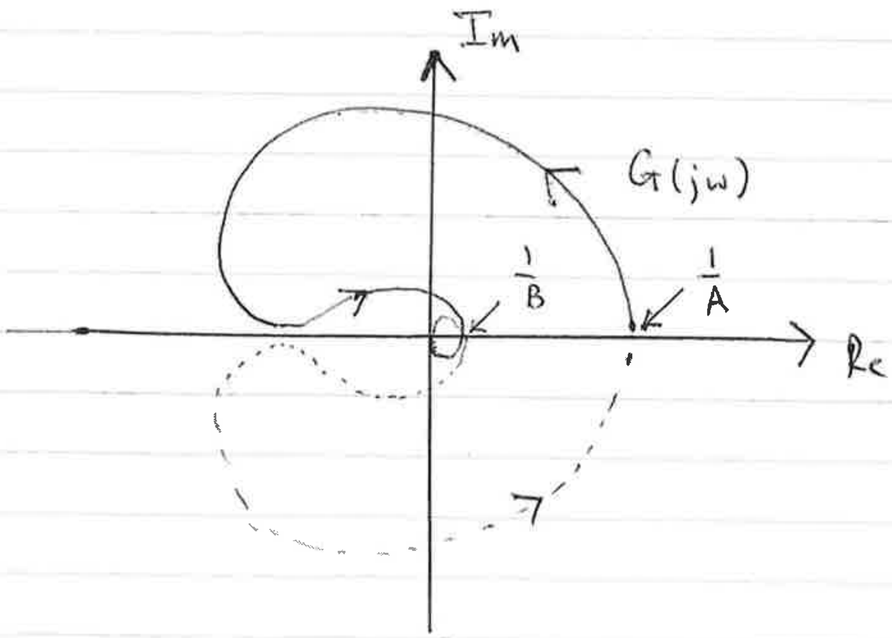
Actual t.f. $\quad G(s) = \dfrac{(s+1)(s-18)^2}{(s^2-2s+2)(s+18)^2}$

$$\Rightarrow \quad B(s) = \frac{s^2+2s+2}{s^2-2s+2}\left(\frac{s-18}{s+18}\right)^2$$

(c) Crossover above 1 rad/sec

below 20 " "

(d)



$k > 0$: 2 RHP poles, $\quad -B < k < -A$: 1 RHP pole

$-A < k < 0$: " " " $\quad k < -B$ 3 " pole

(e) Choose gain cross around peak phase, say 4 rad/sec. Put in a lead compensator to give > 30° PM and to make this the gain crossover frequency. Choose

$$K(s) = k_p \alpha \frac{s + 4/\alpha}{s + 4\alpha}$$

$\alpha = 2 \Rightarrow$ peak phase of $36.87°$ at $w = 4$.
Need to add around 12dB to make $w = 4$ the gain crossover frequency $\Rightarrow k_p \simeq 3.98$.
Try

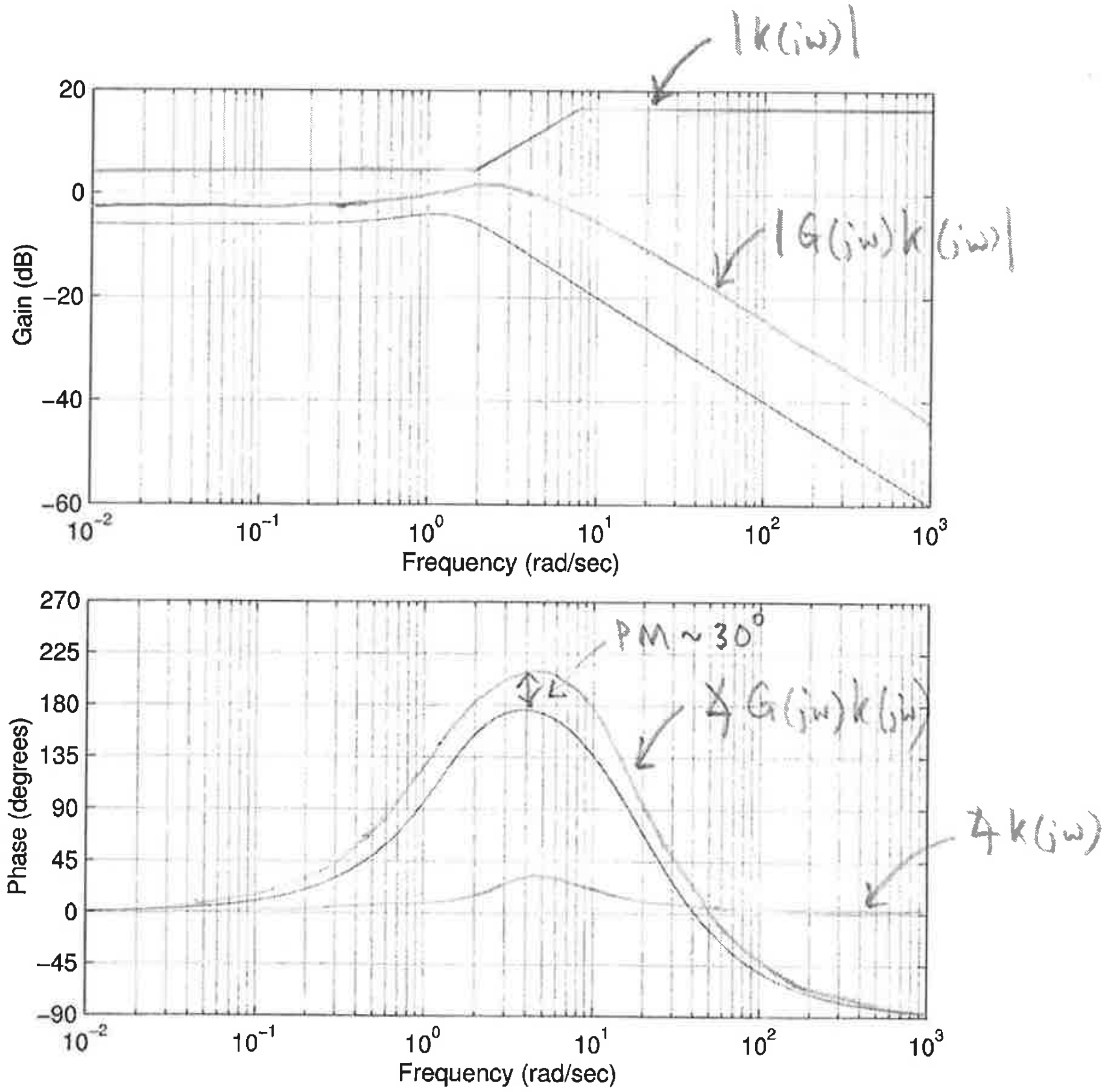
$$K(s) = \frac{4(2s+4)}{s+8} = \frac{8s+16}{s+8}$$

[Note: the plant has slope around -20 dB/dec near $w = 4$ rad/sec, so a lead compensator potentially flattens the magnitude characteristic, but only a small amount of phase lead is required, hence only a short +20dB/dec section in $K(s)$, which doesn't completely flatten the slope. Nevertheless, GM is rather small for this design.]

3(e)

ENGINEERING TRIPOS PART IIB

Friday 13 May 2013, Module 4F1, Question 3.



Extra copy of Fig. 1: Bode diagram of $G(s)$ for Question 3.

3(e) Accurate computer plot with $k(s) = 3.9 \frac{(2s+4)}{s+8}$

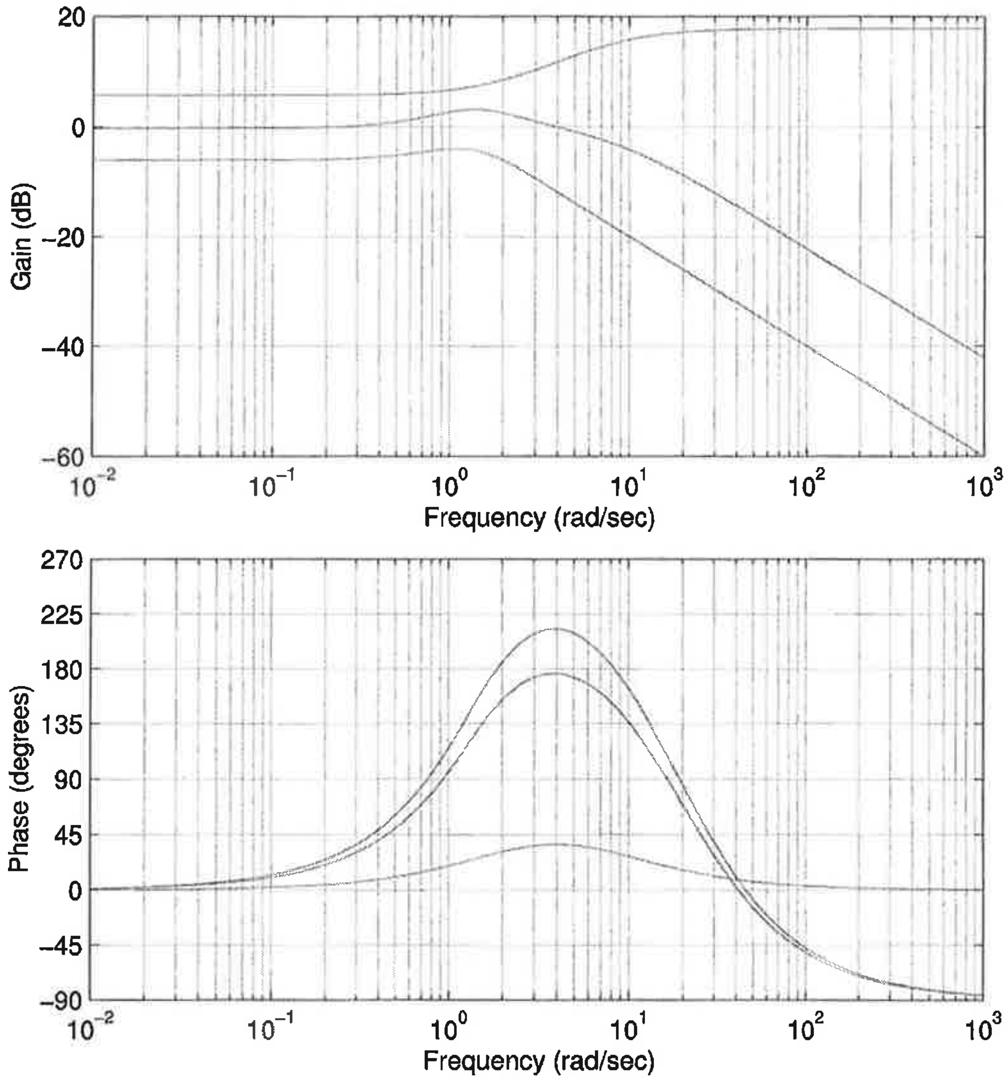


Fig. 2

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