Module 4F2: Solutions 2013

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The solution to Q1 starts on the next page.

1 (a) (i) The $\mathscr{L}_2$-norm of the signal $u(t)$ is defined by

$$||u||_2 = \sqrt{\int_{-\infty}^{\infty} |u(t)|^2 dt}$$

[10%]

(ii) Note that

$$\begin{aligned} ||\hat{y}||_2^2 &= ||G\hat{u}||_2^2 = \int_{-\infty}^{\infty} |G(j\omega)|^2 \, |\hat{u}(j\omega)|^2 \, d\omega \\ &\leq ||G(j\omega)||_\infty^2 \int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 \, d\omega = ||G||_\infty^2 \, ||\hat{u}||_2^2 \\ \Rightarrow \frac{||\hat{y}||_2}{||\hat{u}||_2} &\leq ||G||_\infty \text{ for any } u \neq 0 \end{aligned}$$

To show that maximising LHS gives equality requires a judicious choice of u. Idea: find ω_o where $|G(j\omega)|$ achieves maximum then choose

$$\begin{aligned} u(t) &= \sin \omega_o t \\ \Rightarrow y(t) &\to |G(j\omega_o)| \, . \sin\left(\omega_o t + \angle G(j\omega_o)\right) \\ \Rightarrow \sqrt{\text{energy ratio}} &\to |G(j\omega_o)| \end{aligned}$$

(Technical point: the integral of $u^2(t)$ will $\to \infty$, so we need to take a sinusoid of finite but very long duration).

[40%]

(b) (i) See attached [35%]

(ii) " " [15%]

(b)(i)

$$|G(j\omega)|^2 = \frac{\omega_n^4}{(\omega^2-\omega_n^2)^2+(2c\omega_n\omega)^2} = f(\omega)$$

$$f'(\omega)=0 \iff 0 = \frac{d}{d\omega}\left(\omega^4 - 2\omega^2\omega_n^2 + \omega_n^4 + 4c^2\omega_n^2\omega^2\right)$$

$$= 4\omega^3 - \omega\left(4-8c^2\right)\omega_n^2$$

$$\iff \omega = 0,\ \sqrt{1-2c^2}\,\omega_n = \omega_1$$

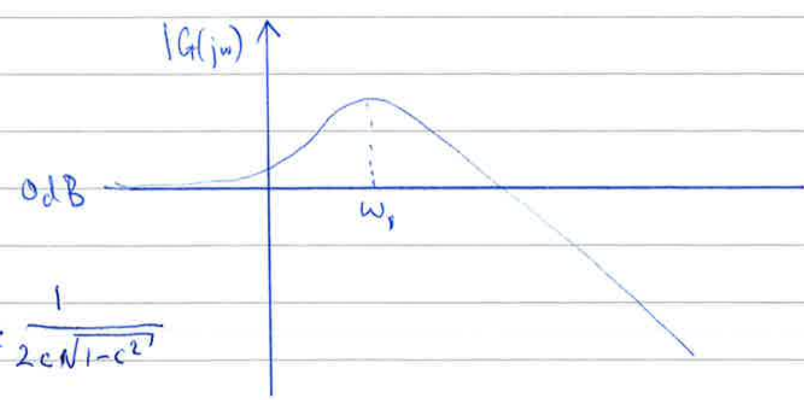
$$(\text{if } c^2 < \tfrac{1}{2})$$

$$|G(j\omega_1)|^2 = \frac{\cancel{\omega_n^4}}{\cancel{\omega_n^4}(1-2c^2-1)^2 + 4c^2(1-2c^2)\cancel{\omega_n^4}}$$

$$= \frac{1}{4c^4 + 4c^2(1-2c^2)} = \frac{1}{4c^2(1-c^2)}$$

$$\Rightarrow |G(j\omega_1)| = \frac{1}{2c\sqrt{1-c^2}} > 1 \quad \text{if } c^2 < \frac{1}{2}$$

$G(0) = 1$



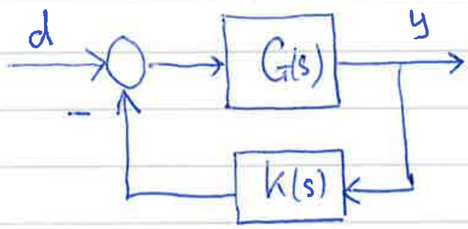
Hence $\|G(s)\|_2 = \frac{1}{2c\sqrt{1-c^2}}$
if $c^2 < \frac{1}{2}$.

b (i) cont.

$$|G(j\omega)|^2 = \frac{\omega_n^4}{\omega^4 + (4c^2 - 2)\omega_n^2\omega^2 + \omega_n^4}$$

Hence $|G(j\omega)|$ achieves its maximum at $\omega = 0$ when $c^2 \geq \frac{1}{2}$. Hence $\|G(s)\|_\infty = 1$ if $c^2 \geq \frac{1}{2}$.

b (ii)



$$T_{d\to y} = \frac{G}{1+GK}$$

Try $K(s) = k$ (proportional gain):

$$T_{d\to y} = \frac{\omega_n^2}{s^2 + 2c\omega_n s + \omega_n^2(1+k)} = G_1$$

Effective damping ratio $c_1 = \dfrac{2c\omega_n}{2\omega_n\sqrt{1+k}} = \dfrac{c}{\sqrt{1+k}}$

So proportional gain reduces damping ratio (makes the system more oscillatory). A proportional + derivative controller (or a phase lead compensator) would be better.

[Answers as above sufficient for full credit.]

Incidentally, note that for $c_1^2 < \frac{1}{2}$

$$\|G_1\|_\infty = \frac{1}{1+k} \cdot \frac{1}{2c_1\sqrt{1-c_1^2}} = \frac{1}{2c\sqrt{1+k-c^2}}$$

so proportional gain does actually reduce the H_∞-norm even though the system becomes more oscillatory in the process.

2 (a) (i) $G = \tilde{M}^{-1}\tilde{N}$ is a left coprime factorisation over H_∞ if $\tilde{M}, \tilde{N}$ belong to H_∞, $\tilde{M}$ being a square matrix, and

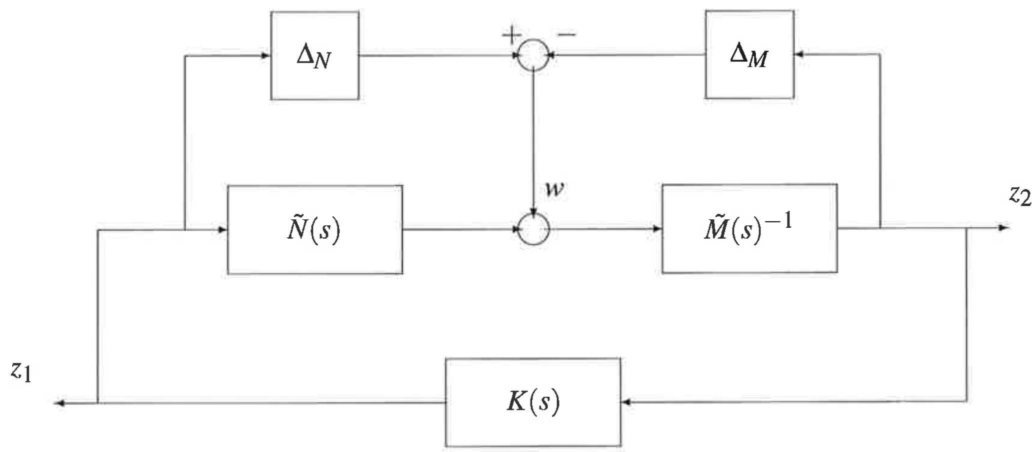
$$\text{rank}\begin{bmatrix}\tilde{N}(s) & \tilde{M}(s)\end{bmatrix} = p \quad \text{for all } s \text{ with } \mathrm{Re}(s) \geq 0 \text{ or } s = \infty$$

The factorisation is normalised if

$$\tilde{M}(j\omega)\tilde{M}(j\omega)^* + \tilde{N}(j\omega)\tilde{N}(j\omega)^* = I \;\text{ for all } \omega$$

[20%]

(ii) Consider the following block diagram:



And we obtain:
$$\begin{aligned} z_2 &= \tilde{M}^{-1}\left(-\Delta_M z_2 + \left(\Delta_N + \tilde{N}\right) z_1\right) \\ \Rightarrow \left(\tilde{M} + \Delta_M\right) z_2 &= \left(\tilde{N} + \Delta_N\right) z_1 \\ \Rightarrow z_2 &= \underbrace{\left(\tilde{M} + \Delta_M\right)^{-1}\left(\tilde{N} + \Delta_N\right)}_{G_\Delta} z_1 \quad \text{as desired.} \end{aligned}$$

Also considering the controlled system gives,

$$\begin{aligned} z_2 &= \tilde{M}^{-1}\left\{w + \tilde{N}K z_2\right\} \\ \left(I - \tilde{M}^{-1}\tilde{N}K\right) z_2 &= \tilde{M}^{-1} w \\ z_2 &= (I - GK)^{-1}\tilde{M}^{-1} w \\ z_1 &= K z_2 \\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1}\tilde{M}^{-1} w \\ w &= [\Delta_N, -\Delta_M]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned}$$

Hence it follows that the above closed loop is internally stable for all $\|[\Delta_N, \Delta_M]\|_\infty < \varepsilon$

$$\Leftrightarrow \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \tilde{M}^{-1} \right\|_\infty \leq 1/\varepsilon, \quad \text{(by the Small Gain Theorem)}.$$

Therefore the closed loop system remains stable for all such perturbations if and only if

$$b(G,K) = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \tilde{M}^{-1} \right\|_\infty^{-1} \geq \varepsilon.$$

[25%]

(b) (i) See attached [25%]

(ii) " " [30%]

(b) (i)

$$G(s) = \frac{3}{s+1} \Big/ \frac{s-4}{s+1} \qquad \text{coprime}$$

$$N^*N + M^*M = \frac{3}{-s+1}\,\frac{3}{s+1} + \frac{-s-4}{-s+1}\,\frac{s-4}{s+1}$$

$$= \frac{-s^2+25}{-s^2+1} = \left(\frac{s+5}{s+1}\right)^{-1} \frac{s+5}{s+1}$$

$$\Rightarrow \quad G(s) = \frac{3}{s+1}\,\frac{s+1}{s+5} \Big/ \frac{s-4}{s+1}\,\frac{s+1}{s+5}$$

$$= \frac{3}{s+5} \Big/ \frac{s-4}{s+5} \qquad \text{is a normalised coprime factorisation}$$

(ii)

$$\left\| \begin{pmatrix} K \\ I \end{pmatrix} (I - GK)^{-1} \tilde{M}^{-1} \right\|_\infty = \left\| \begin{pmatrix} -2 \\ 1 \end{pmatrix} \frac{1}{1 + \frac{6}{s-4}}\,\frac{s+5}{s-4} \right\|_\infty$$

$$= \left\| \begin{pmatrix} -2 \\ 1 \end{pmatrix} \frac{s+5}{s+2} \right\|_\infty = \sqrt{5} \left\| \frac{s+5}{s+2} \right\|_\infty = \sqrt{5} \cdot \frac{5}{2}$$

From Bode diagram, max. achieved at $s=0$

Hence $b(G,k) = \frac{2}{5\sqrt{5}}$

3. (a) A sector nonlinearity is a function $f(.)$ such that $\alpha e \le ef(e) \le \beta e$ for some constants α and β. The function is then said to be in the sector (α, β), also denoted by $\alpha \le f(.) \le \beta$.

(b) Circle criterion for the system shown in Fig.1: If $\alpha \le f(.) \le \beta$ then the closed loop is globally asymptotically stable (with unique equilibrium at the origin) if the Nyquist locus of $G(s)$ encircles the circle $D(-1/\beta, -1/\alpha)$ as many times counter-clockwise as $G(s)$ has unstable poles, where $D(-1/\beta, -1/\alpha)$ is the circle with $-1/\beta$ and $-1/\alpha$ as end-points of a diameter.

(c) Standard bookwork. Details are in Handout 4 of the 4F2 course. The solution should preferably include a block diagram showing the transformations performed on the linear system and on the nonlinear gain.

(d) In this case we need to apply the Circle Criterion with $\alpha = -k < 0$ and $\beta = k > 0$. As $\alpha \to 0$, the circle of the criterion becomes a half-plane through $-1/\beta$, and as α becomes negative, what was previously the interior of the circle now becomes its exterior. Thus the criterion requires the Nyquist locus of $G(s)$ to lie entirely *inside* the circle $D(-1/k, 1/k)$. This means that the Nyquist locus cannot encircle the circle, which is ok because $G(s)$ has no unstable poles. It can be seen (with the aid of a compass) that the Nyquist locus shown in Fig.2 lies entirely within the circle $D(-4, 4)$ but not for any value of $1/k$ smaller than 4 — see figure below. Thus the largest value of k for which global asymptotic stability is assured by the Circle Criterion is $\boxed{k = 1/4}$.

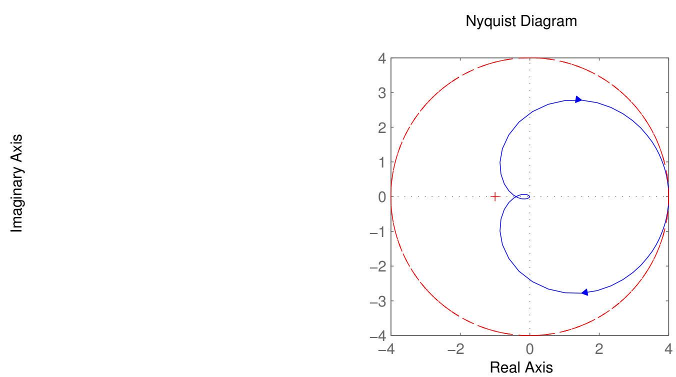


Figure 1: Nyquist locus inside circle $D(-4, 4)$.

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4. (a) LaSalle's Theorem: Let $S \subseteq \Re^n$ be a compact invariant set. Assume there exists a differentiable function $V : S \to \Re$ such that

$$\dot{V}(x) \le 0 \; \forall x \in S$$

Let M be the largest invariant set contained in $\{x \in S \mid \dot{V}(x) = 0\}$ (the set of $x \in S$ for which $\dot{V}(x) = 0$). Then all trajectories starting in S approach M as $t \to \infty$.
Corollary: If the set $\{x \in S \mid \dot{V}(x) = 0\}$ contains no trajectories other than $x(t) = 0$, then 0 is locally asymptotically stable. Moreover, all trajectories starting in S converge to 0.

(b) When using Lyapunov's theorems to establish asymptotic stability of an equilibrium $x = 0$, it is necessary to show that $\dot{V}(x) = 0$ at $x = 0$ only. In many practical examples this is not true but the conditions of LaSalle's theorem, which are weaker, *are* satisfied. LaSalle's Theorem can also be applied to establish convergence to more general invariant sets, such as limit cycles.

(c) Let $x_1 = x$, $x_2 = \dot{x}$. Then the equation of motion becomes

$$\begin{aligned} \dot{x}_1 &= x_2 && (1) \\ \dot{x}_2 &= -\frac{c(x_1)}{m} - \frac{b(x_2)}{m} && (2) \end{aligned}$$

At an equilibrium we need $(\dot{x}_1, \dot{x}_2) = (0.0)$. $\dot{x}_1 = 0 \Rightarrow x_2 = 0$. But $b(0) = 0$, so for $\dot{x}_2 = 0$ we need $c(x_1) = 0$, which occurs only for $x_1 = 0$. Thus the equilibrium is (0.0), and it is unique. (The fact that it is unique could also be deduced from part (d), which asks for global asymptotic stability to be proved. Global asymptotic stability is only possible if there is a unique equilibrium.)

(d) We have

$$V(x_1, x_2) = \frac{mx_2^2}{2} + \int_0^{x_1} c(\nu)d\nu \tag{3}$$

Clearly $V(0,0) = 0$. A bit less obviously, $\int_0^{x_1} c(\nu)d\nu > 0$. If $x_1 > 0$ this is clear, since $\nu c(\nu) > 0$. If $x_1 < 0$ it also holds, since the increments $d\nu$ are negative in this case. Consequently, $V(x_1, x_2) > 0$ if $(x_1, x_2) \neq (0,0)$. V is thus a candidate Lyapunov function, and we investigate its derivative along system trajectories:

Assuming $\dot{x} = f(x)$ we have

$$\begin{aligned} \dot{V}(x) &= \nabla V(x)^T f(x) && (4) \\ &= [c(x_1), mx_2] \begin{bmatrix} x_2 \\ -\frac{c(x_1)}{m} - \frac{b(x_2)}{m} \end{bmatrix} && (5) \\ &= -x_2 b(x_2) < 0 \quad \text{if} \quad x_2 \neq 0. && (6) \end{aligned}$$

This is enough to prove stability of $(0,0)$, but not asymptotic stability since $\dot{V} = 0$ whenever $x_2 = 0$ even if $x_1 \neq 0$. We need to invoke LaSalle's theorem.

Suppose that, at some time t, $x_1(t) \neq 0, x_2(t) = 0$. Then $\dot{x}_2(t) = -c(x_1)/m \neq 0$. Thus there exists a $\tau > 0$ such that $x_2(t+\tau) \neq 0$, and hence $\dot{V}(x(t+\tau)) < 0$. So the only invariant set $\{x : \dot{V}(x) = 0\}$ is $(0,0)$. Furthermore, the set $\{x : V(x) \leq V(x_0)\}$ is an invariant set for any initial condition x_0 (since $\dot{V} \leq 0$), and it contains $(0,0)$. Thus a trajectory starting at x_0 will converge to $(0,0)$. But all of this holds, no matter how large $\|x_0\|$ may be. Therefore $(0,0)$ is globally asymptotically stable.