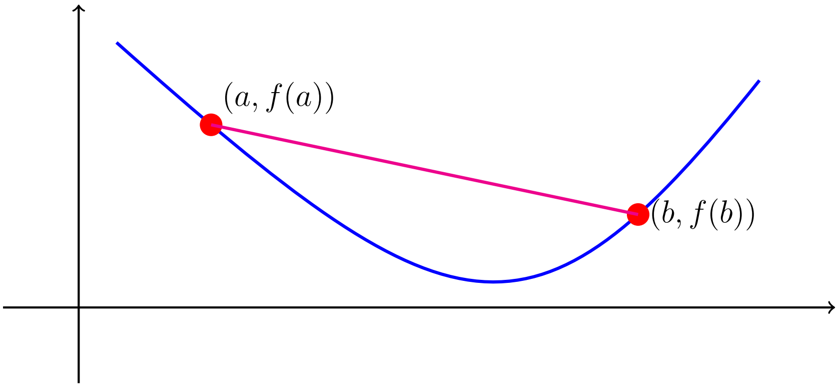
Part IIB Paper 4F3 Exam Paper Sample Solutions 2013

1. **Solution:**

   (a) i. The function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $a, b \in \mathbb{R}^n$ and for all $\beta \in [0,1]$.

   $$f(\beta a + (1-\beta)b) \leq \beta f(a) + (1-\beta)f(b)$$

   For $n = 1$ this implies that the curve is always beneath the chord:



   ii. A necessary and sufficient condition for f to be a convex function is that its Hessian matrix is positive semi-definite. If y is fixed then $g(x) = x^2+y^2+4xy$ has Hessian, $\frac{d^2g}{dx^2} = 2 > 0$ and hence g is convex for any value of y. Similarly for x fixed. However the Hessian matrix for $f(x,y) = x^2+y^2+4xy$ is $H = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ and this is not positive semi-definite since e.g. its determinant is -12. [Aside: Also note that $f(x,-x) = -2x^2$ so with $a = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $b \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ we have $f(a) = 0$, $f(b) = -8$ and with $\beta = 0.5$

   $$f(\beta a+(1-\beta)b) = f(1,-1) = -2 > \beta f(a)+(1-\beta)f(b) = \frac{1}{2}(0-8) = -4$$

   and hence f is not a convex function.]

   (b) i. The equality constraints determine the relationship between the predicted inputs and the predicted states. The state prediction equations are:

   $$x_1 = Ax_0 + Bu_0$$
   $$x_2 = Ax_1 + Bu_1$$
   $$x_3 = Ax_2 + Bu_2.$$

If we re-arrange these

$$
\begin{aligned}
-Ax_0 - Bu_0 + x_1 &= 0\\
-Ax_1 - Bu_1 + x_2 &= 0\\
-Ax_2 - Bu_2 + x_3 &= 0,
\end{aligned}
$$

and rewrite in matrix form, we get:

$$
\underbrace{\begin{bmatrix} -B & I & & & & \\ 0 & -A & -B & I & & \\ 0 & 0 & 0 & -A & -B & I \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ x_2 \\ u_2 \\ x_3 \end{bmatrix}}_{\underline{\theta}} - \underbrace{\begin{bmatrix} Ax_0 \\ 0 \\ 0 \end{bmatrix}}_{f} = 0.
$$

Obviously, if both F and f are both multiplied by -1, this would be equally correct.

[A number of candidates chose to unnecessarily substitute for x_1, x_2, x_3 in the terms Ax_i complicating the expressions for F and f.]

The matrix H is trivially:

$$
\begin{bmatrix} R & & & & & \\ & Q & & & & \\ & & R & & & \\ & & & Q & & \\ & & & & R & \\ & & & & & P \end{bmatrix}
$$

ii. $R \succeq 0$, $Q \succeq 0$, $P \succeq 0$ are sufficient conditions for a convex objective, $H \succeq 0$, and if in addition $R \succ 0$ then the solution will be unique from results on the linear quadratic regulator.

iii.

$$
\underline{\mathbf{y}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C \end{bmatrix}}_{\mathcal{C}_e} \underline{\theta}
$$

$$
\underline{\mathbf{y}}_{\max} = \begin{bmatrix} y_{\max} \\ y_{\max} \\ y_{\max} \end{bmatrix}
$$

So, the optimisation in standard form is:

$$
\begin{aligned}
\text{Minimize:} \quad & \underline{\theta}^T H \underline{\theta}\\
\text{Subject to:} \quad & F\underline{\theta} - f = 0\\
\text{and} \quad & \begin{bmatrix} \mathcal{C}_e \\ -\mathcal{C}_e \end{bmatrix} \underline{\theta} - \begin{bmatrix} \underline{\mathbf{y}}_{\max} \\ \underline{\mathbf{y}}_{\max} \end{bmatrix} \leq 0
\end{aligned}
$$

Alternatively, (and possibly more convenient from the point of view of exploiting the structure of the problem in a solution algorithm),

$$\underline{\hat{\mathbf{y}}} = \begin{bmatrix} y_1 \\ -y_1 \\ y_2 \\ -y_2 \\ y_3 \\ -y_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & C & 0 & 0 & 0 & 0 \\ 0 & -C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & -C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 & 0 & -C \end{bmatrix}}_{\hat{C}_e} \underline{\theta}$$

$$\underline{\hat{\mathbf{y}}}_{\max} = \begin{bmatrix} y_{\max}^T & y_{\max}^T & y_{\max}^T & y_{\max}^T & y_{\max}^T & y_{\max}^T \end{bmatrix}^T.$$

Then the problem can be cast as:

$$\begin{aligned} \text{Minimize:} \quad & \underline{\theta}^T H \underline{\theta} \\ \text{Subject to:} \quad & F\underline{\theta} - f = 0 \\ \text{and} \quad & \hat{C}_e \underline{\theta} - \underline{\hat{\mathbf{y}}}_{\max} \leq 0. \end{aligned}$$

iv.
- The new constraint on u_k is non-convex, so the optimisation problem is no longer a convex problem, and is substantially more difficult to solve since it is no longer guaranteed that a local optimum is the global optimum.
  - Can form as a mixed integer quadratic program for which effective (but still exponentially complex in the worst case) solution algorithms exist (not on syllabus)
  - Computation of the exact solution may take a very long time (the worst case problem complexity grows exponentially with the number of the non-convex constraints). Although in this case there are only 3 inputs to determine so that this is equivalent to $3^3 = 27$ convex optimization problems, which might be quite feasible.
  - A suboptimal solution might have to be accepted if computation takes too long.
- The discontinuous input constraint means that if the plant is not open-loop stable, the closed-loop trajectory could end up in a limit cycle rather than at the origin.

[Rather few attempts correctly discussed the non-convexity of the allowable set of u and the resulting computational complexity.]

2. **Solution:**

(a) i. The receding horizon principle:

1. Sample current system state

2. Optimise a finite length sequence of future plant inputs, subject to constraints on the inputs, and predicted future states (and outputs).
3. Apply the first element of the sequence to the plant.
4. Discard the prediction, and wait until the next time step
5. Go to step 1.

The prediction horizon remains the same length, but starts at a point in time 1 step into the future with respect to the original horizon. Hence, the horizon recedes.

A receding horizon controller is a feedback controller, since it acts on new information at each time step: at each time step the current state measurement (estimate) is used to calculate a new optimal control sequence and only the first element of this is applied.

ii. Merits:

- Handles MIMO plants naturally
- Optimisation in the loop can improve performance by reducing variability in outputs
- Can handle constraints
- Can handle general nonlinearities - at least such problems can be posed
- Constrained optimisation enables the plant to operate closer to the edge of the feasible region, which can have economic benefits
- Receding horizon is a computationally tractable way of approximately solving an otherwise very difficult infinite horizon control problem (e.g. with nonlinearities and constraints)
- Problem formulation in terms of "what" should be achieved by the controller rather than "how" it should be achieved
- Can be intuitively reconfigured online (e.g. changing constraints, updating plant model) to accommodate changes in circumstances (e.g. varying plant dynamics, or faults).
- Can compute explicit solution (for small problems)

Drawbacks:

- A good quality model of the plant is necessary
- Numerical optimisation is more computationally demanding than state feedback
- Certification of correctness of implementation of an iterative optimisation algorithm is more difficult than certification of an analytical feedback policy
- The cost function in the optimisation problem can be difficult to tune
- Assumptions necessary for formal proof of stability can be restrictive
- The future disturbances and reference signals need to be assumed or estimated.

- In the constrained case if the quadratic programme has no feasible solution then the method fails and some alternative scheme needs to be determined in real time.

(b) i. We note that:

- The disturbance is perfectly measured and included in the prediction model;
- There is no plant-model mismatch;
- With no disturbance, the controller would be stabilising (due to the choice of P);
- The plant is fully reachable/controllable;
- The matrix $Q > 0$, so we cannot blame non-observability or non-detectability of the the pair $(Q^{1/2}, A)$.

The constant disturbance d means that the origin is not an unforced equilibrium — i.e. to stay at the origin a non-zero values of u is required. However, the cost function with $x_s = 0$ and $u_s = 0$ is minimising a weighted quadratic function of the deviation of the state (and therefore also the controlled output) and the input from zero. Since the cost function is quadratic, the minimiser achieves a compromise between deviation of u from zero and deviation of x from zero.

ii. We note that:

- The values Q, R and P might be changed in the future, so we cannot simply find the analytical solution $u = Kx$ and find a point x_s so that $u = K(0 - x_s)$ balances the disturbance;
- The structure of the cost function may not be changed, so reformulating in terms of Δu is not allowed, neither is using an exact penalty cost function.

The pair (x_s, u_s) should be chosen to be a target equilibrium pair satisfying:

$$\begin{aligned} Ax_s + Bu_s + B_d d &= x_s \\ Hx_s &= 0. \end{aligned}$$

Note that we then have,

$$\begin{aligned} (x_{k+1} - x_s) &= A(x_k - x_s) + B(u_k - u_s) \\ z_k &= H(x_k - x_s) \end{aligned}$$

and stability of the resulting MPC law (due to the choice of P) gives $x_k \to x_s$ and $u_k \to u_s$ as $k \to \infty$.

iii. The new pair (x_s, u_s) should be chosen to be a target equilibrium pair satisfying:

$$\begin{aligned} Ax_s + Bu_s + B_d d &= x_s \\ Hx_s &= r. \end{aligned}$$

iv. By rearranging the previously states equations, the pair (x_s, u_s) has to satisfy:

$$\begin{bmatrix} (A-I) & B \\ H & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} -B_d d \\ r \end{bmatrix}$$

A solution to this set of equations for any r will exist if the matrix (Note that $(I-A)^{-1}$ exists since $\rho(A) < 1$).

$$\begin{bmatrix} (A-I) & B \\ H & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ H(A-I)^{-1} & I \end{bmatrix} \begin{bmatrix} (A-I) & B \\ 0 & H(I-A)^{-1}B \end{bmatrix}$$

has full row rank. Hence we require that the steady-state gain, $G(1) = H(I-A)^{-1}B$ to have full row rank which implies that there needs to be as many inputs as outputs.

[None of the attempts cited the form of P to deduce stability of the equilibrium.]

3. **Solution:**

(a) $G(s) = C\,(sI - A)^{-1}\,B$ has a state-space realisation,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

and balanced realisations are defined for stable systems so we need the eigen values of A to be strictly in the left half of the complex plane. The controllability Gramian, P, satisfies the Lyapunov equation,

$$AP + PA^T + BB^T = 0$$

and the observability Gramian, Q, satisfies

$$A^T Q + QA + C^T C = 0$$

If the system is stable, controllable and observable then $P > 0$ and $Q > 0$. A change of state-space coordinates exists such that $P = Q = \Sigma$ in these new coordinates where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are called the Hankel singular values. It we write the balanced realisation as

$$\left[\begin{array}{cc} A & B \\ C & 0 \end{array}\right] = \left[\begin{array}{ccc} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & 0 \end{array}\right] \quad \text{where} \quad A_{11} \in \mathbb{R}^{k\times k} \quad \text{etc}$$

the reduced order truncated balanced realisation, satisfies

$$\left\| C\,(sI - A)^{-1}\,B - C_1\,(sI - A_{11})^{-1}\,B_1 \right\|_\infty \leq 2 \times (\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_n)$$

Hence if σ_{k+1} is small compared to, for example $\|G(s)\|_\infty$, then this gives a good approximate model.

(b) i. The given state equations conform to those given in the attached Data Sheet with identical notation. The state feedback controller is given by $u_{opt} = -Fx$ where $F = B_2^T X$ and X solves the CARE with $A - B_2 F$ a stable matrix. Also the derivation in lectures has assumed that (A, B_2) is controllable and (A, C_1) is observable which are true since $[B_2 \ AB_2] = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} C_2 \\ C_2 A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ both of which are clearly rank 2. We need to first check that the given $X = \begin{bmatrix} \pm 1 & 1 \\ 1 & \pm 2 \end{bmatrix}$ satisfy the CARE:

$$
\begin{aligned}
A^T X + XA + C_1^T C_1 - XB_2 B_2^T X &= \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \pm 1 & 1 \\ 1 & \pm 2 \end{bmatrix} + \\
&+ \begin{bmatrix} \pm 1 & 1 \\ 1 & \pm 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ \pm 2 \end{bmatrix} \begin{bmatrix} 1 & \pm 2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark
\end{aligned}
$$

The stability of $(A - B_2 F)$ can be checked directly or equivalently by checking $X > 0$. Clearly only $X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} > 0$, and then $F = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $A - B_2 F = \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix}$ whose eigen values satisfy $\lambda^2 + 2\lambda + 2 = 0$ and $\lambda_{1,2} = -1 \pm j$ which verifies stability, and give the corresponding closed loop poles.

ii. The output feedback case also fits the form given in the Data Sheet. The derivation given in lectures also assumed that (A, B_1) is controllable and (A, C_2) is observable. Since $C_2 = C_1$ we have observability but $[B_1 \ AB_1] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ which has rank 1 and is not controllable. However let's check that the FARE is satisfied with $Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ when:

$$
\begin{aligned}
AY + YA^T + B_1 B_1^T - YC_2^T C_2 Y &= \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \\
&+ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark
\end{aligned}
$$

However Y is not positive definite, but just positive semidefinite, so let's check the eigen values of $A - HC_2$ where $H = YC_2^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A - HC_2 = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$ with eigen values at 0 and -1, so there is an unstable eigen value at $\lambda = 0$. This controller will hence produce closed-loop poles at $-1 \pm j,\ -1,\ 0$. The controller will have transfer function,

$$\begin{aligned}
K(s) &= -F\left(sI - A + B_2F + HC_2\right)^{-1} H \\
&= -\left[\begin{array}{cc} 1 & 2 \end{array}\right] \left[\begin{array}{cc} s+1 & -2 \\ 1 & s+2 \end{array}\right]^{-1} \left[\begin{array}{c} 1 \\ 0 \end{array}\right] \\
&= -\frac{1}{s^2+3s+4} \left[\begin{array}{cc} 1 & 2 \end{array}\right] \left[\begin{array}{c} s+2 \\ -1 \end{array}\right] \\
&= -\frac{s}{s^2+3s+4}
\end{aligned}$$

The controller has therefore cancelled the plant pole at 0 because this plant pole is not excited by the disturbance w_1. However this results in this pole not being stabilised by the controller and hence any initial condition would not decay to zero.

[Very few attempts considered any of the controllability or observability conditions.]

4. **Solution:**

(a) The $\mathcal{H}_\infty$ norm of a transfer function $G(s)$ is defined for $G(s) \in \mathcal{H}_\infty$, i.e. all the poles of $G(s)$ have real part < 0, and hence the system is stable, when

$$\|G(s)\|_\infty = \sup_{-\infty<\omega<\infty} \sigma_{\max}\left(G(j\omega)\right) \tag{1}$$

where $\sigma_{\max}$ denotes the largest singular value. This gives the maximum gain from the input to the output, i.e. if $\bar{y}(s) = G(s)\bar{u}(s)$ then,

$$\|y\|_2 \leq \|G(s)\|_\infty \|u\|_2$$

Hence in assessing system performance if w represent disturbances and z represent errors to be minimised in the face of the disturbances, then $\|G\|_\infty$ gives the maximum (i.e. worst case) amplification of the disturbances. This can also be interpreted on frequency-by-frequency basis. (In robust control the small gain theorem and its variants give robust stability conditions in the face of plant uncertainty, but this was not specifically covered in 4F3).

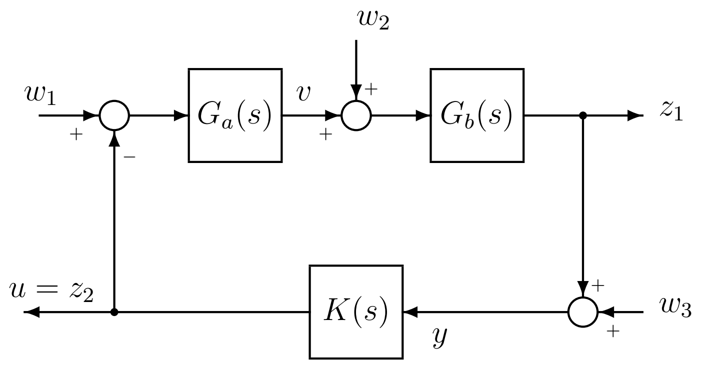
To calculate $\|G\|_\infty$ one method is to take a grid of frequencies, ω_i, and calculate $\max_i \sigma_{\max}(G(j\omega_i))$ which gives a lower bound on $\|G\|_\infty$ and will give a good estimate of $\|G\|_\infty$ with a fine grid. Alternatively in state-space $\|G\|_\infty < \gamma$ if (and only if) the algebraic Riccati equation,

$$A^TX + XA + C^TC + \gamma^{-2}XBB^TX = 0$$

has a solution, $X > 0$. A bisection search on γ can then be performed to obtain $\|G\|_\infty$ to any desired accuracy.

[Almost no attempts cited the Riccati equation method for calculating this norm.]

(b) Consider the block diagram:



i. We wish to find a $P(s)$ such that

$$\begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{y} \end{bmatrix} = P(s) \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \bar{w}_3 \\ \bar{u} \end{bmatrix}$$

$$\begin{aligned}
\bar{z}_1 &= G_b\left(\bar{w}_2 + G_a(\bar{w}_1 - \bar{u})\right) \\
\bar{z}_2 &= \bar{u} \\
\bar{y} &= \bar{w}_3 + \bar{z}_1 = \bar{w}_3 + G_b\left(\bar{w}_2 + G_a(\bar{w}_1 - \bar{u})\right) \\
\begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{y} \end{bmatrix} &= \underbrace{\begin{bmatrix} G_bG_a & G_b & 0 & -G_bG_a \\ 0 & 0 & 0 & I \\ G_bG_a & G_b & I & -G_bG_a \end{bmatrix}}_{P(s)} \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \bar{w}_3 \\ \bar{u} \end{bmatrix}
\end{aligned}$$

[A number of attempts started eliminating y or u from the equations so missing the point of this formulation.]

ii. The state equations for P are given by:

$$\begin{aligned}
\dot{x}_a &= A_a x_a + B_a(w_1 - u), \quad v = C_a x_a \\
\dot{x}_b &= A_b x_b + B_b(w_2 + v) = A_b x_b + B_b C_a x_a + B_b w_2 \\
z_1 &= C_b x_b \\
z_2 &= u \\
y &= w_3 + C_b x_b
\end{aligned}$$

and combining these gives:

$$\left[\begin{array}{c}\dot{x}_a\\ \dot{x}_b\\ \hline z_1\\ z_2\\ y\end{array}\right] = \left[\begin{array}{cc|cccc} A_a & 0 & B_a & 0 & 0 & -B_a\\ B_bC_a & A_b & 0 & B_b & 0 & 0\\ \hline 0 & C_b & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & I\\ 0 & C_b & 0 & 0 & I & 0\end{array}\right]\left[\begin{array}{c}x_a\\ x_b\\ \hline w_1\\ w_2\\ w_3\\ u\end{array}\right]$$

$$= \left[\begin{array}{c|ccc} A & B_1 & 0 & B_2\\ \hline C_1 & 0 & 0 & 0\\ 0 & 0 & 0 & I\\ C_2 & 0 & I & 0\end{array}\right]\left[\begin{array}{c}x\\ \hline w\\ u\end{array}\right]$$

where

$$A = \left[\begin{array}{cc} A_a & 0\\ B_bC_a & A_b\end{array}\right],\ B_1 = \left[\begin{array}{cc} B_a & 0\\ 0 & B_b\end{array}\right],\ B_2 = \left[\begin{array}{c} -B_a\\ 0\end{array}\right],$$

$$C_1 = \left[\begin{array}{cc} 0 & C_b\end{array}\right],\ C_2 = C_1.$$

iii. The form of the state equation for $P(s)$ is the same as in the Data Sheet with (A, B_1, B_2, C_1, C_2) as defined. Controllability of (A, B_1) and (A, B_2) is not assured since could have a pole/zero cancellation between G_a and G_b and if this were an unstable pole that would give a problem. Similarly for observability. It is known that $\gamma > \min_{\text{stabilizing}K} \|\mathcal{F}_\ell(P, K)\|_\infty$ if there exist stabilizing solutions to the two ARE's

$$\begin{aligned} XA + A^TX + C_1^TC_1 - X(B_2B_2^T - \gamma^{-2}B_1B_1^T)X &= 0\\ YA^T + AY + B_1B_1^T - Y(C_2^TC_2 - \gamma^{-2}F^TF)Y &= 0\\ F &= B_2^TX\\ X &> 0\\ Y &> 0\end{aligned}$$

A bisection search to find the minimum value of γ can be performed at each stage checking whether the above conditions are satisfied or not. A resulting controller is then as given in the data sheet.

K Glover/E N Hartley, May 2013

Module 4F3 2013 Exam Answers

1.(b)(i)

$$F = \begin{bmatrix} -B & I & & & & \\ 0 & -A & -B & I & & \\ 0 & 0 & 0 & -A & -B & I \end{bmatrix}, f = \begin{bmatrix} Ax_0 \\ 0 \\ 0 \end{bmatrix}, H = \begin{bmatrix} R & & & & & \\ & Q & & & & \\ & & R & & & \\ & & & Q & & \\ & & & & R & \\ & & & & & P \end{bmatrix}$$

(b)(ii) $R \succ 0$, $Q \succeq 0$, $P \succeq 0$ for convex problem with unique solution.

(b)(iii)

$$\hat{\mathcal{C}}_e \underline{\theta} - \hat{\underline{\mathbf{y}}}_{\max} \leq 0, \text{ where } \hat{C}_e = \begin{bmatrix} 0 & C & 0 & 0 & 0 & 0 \\ 0 & -C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & -C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 & 0 & -C \end{bmatrix}, \quad \hat{\underline{\mathbf{y}}}_{\max} = \begin{bmatrix} y_{\max} \\ y_{\max} \\ y_{\max} \\ y_{\max} \\ y_{\max} \\ y_{\max} \end{bmatrix}.$$

2.

(b)(ii/iii)

$$\begin{bmatrix} (A-I) & B \\ H & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} -B_d d \\ r \end{bmatrix}$$

(b)(iv) Need $G(1) = H(I-A)^{-1}B$ to have full row rank which implies that there needs to be as many inputs as outputs.

3.

(b)(i) $X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 2 \end{bmatrix}$, closed-loop poles are $-1 \pm j$.

(b)(ii) $H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_i(A - HC_2) = 0,\ -1$ and hence the closed loop is not asymptotically stable.

4.

(b)(i)

$$\begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{y} \end{bmatrix} = \underbrace{\begin{bmatrix} G_bG_a & G_b & 0 & -G_bG_a \\ 0 & 0 & 0 & I \\ G_bG_a & G_b & I & -G_bG_a \end{bmatrix}}_{P(s)} \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \bar{w}_3 \\ \bar{u} \end{bmatrix}$$

(b)(ii)

$$\left[\begin{array}{c} \dot{x}_a \\ \dot{x}_b \\ \hline z_1 \\ z_2 \\ y \end{array}\right] = \left[\begin{array}{cc|cccc} A_a & 0 & B_a & 0 & 0 & -B_a \\ B_bC_a & A_b & 0 & B_b & 0 & 0 \\ \hline 0 & C_b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & C_b & 0 & 0 & I & 0 \end{array}\right] \left[\begin{array}{c} x_a \\ x_b \\ \hline w_1 \\ w_2 \\ w_3 \\ u \end{array}\right]$$

KG May 2013