Question 1

(a) $$\min \; E\left\{ \left(x(n) - \sum_{i=1}^{L} h_i \, x(n-i) \right)^2 \right\}$$

with respect to $(h_1, \dots, h_L)$.

In standard notation:

$\cancel{d(n) = x(n)}$

$$E\left\{ \left(x(n) - \underline{h}^T \underline{x}(n-1) \right)^2 \right\}$$

where $\underline{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_L \end{bmatrix}$, $\underline{x}(n-1) = \begin{bmatrix} x(n-1) \\ \vdots \\ x(n-L) \end{bmatrix}$

The cost function simplifies to

$$E\left\{ \left((\underline{a} - \underline{h})^T \underline{x}(n-1) + w(n) \right)^2 \right\}$$

(where $\underline{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_L \end{bmatrix}$

$$= (\underline{a} - \underline{h})^T E\left\{ \underline{x}(n-1) \underline{x}(n-1)^T \right\} (\underline{a} - \underline{h}) + E\left\{ w(n)^2 \right\}$$

minimiser is obviously $\underline{h}^* = \underline{a}$

(b) Let $J(\underline{h}) = E\left(\left(\overset{x(n)}{\cancel{d(n)}} - \underline{h}^T \underline{x}(n-1) \right)^2 \right)$

~~where~~ / ~~d(n)~~

Gradient is

$$\nabla J(\underline{h}) = E\left(-2 \left(x(n) - \underline{h}^T \underline{x}(n-1) \right) \underline{x}(n-1) \right)$$

Steepest descent;

$$\underline{h}(n+1) = \underline{h}(n) - \frac{\mu}{2} \nabla J(\underline{h}) \Big|_{\underline{h} = \underline{h}(n)}$$

(c) LMS replaces $\nabla J(\underline{h}) \big|_{\underline{h} = \underline{h}(n)}$ with an ~~ist~~ instantaneous estimate

$$\underline{h}(n+1) = \underline{h}(n) + \mu \left(x(n) - \underline{h}^T(n) \underline{x}(n-1) \right) \underline{x}(n-1)$$

(d)(i) The LMS algorithm is as in part (c) but replace

$x(n)$ with $u(n)$

and $\underline{x}(n-1)$ with $\underline{u}(n-1) = \begin{bmatrix} u(n-1) \\ \vdots \\ u(n-L) \end{bmatrix}$

Follow standard derivation in lecture notes to get

$$E\{\underline{h}(n)\} \rightarrow \underline{R}^{-1}\underline{p} \quad \text{where } \underline{R} = E\{\underline{u}(n-1)\underline{u}(n-1)^T\}$$

$$\underline{p} = E\{u(n)\, \underline{u}(n-1)\}$$

(ii) Note $\underline{p} = E\{x(n)\,\underline{u}(n-1)\} + \underbrace{E\{v(n)\,\underline{u}(n-1)\}}_{=0}$

$$= E\{x(n)\,(\underline{x}(n-1) + \underline{v}(n-1))\}$$

$$= E\{x(n)\,\underline{x}(n-1)\}$$

$$\underline{R} = E\{\underline{u}(n-1)\,\underline{u}(n-1)^T\}$$

$$= E\{\underline{x}(n-1)\,\underline{x}(n-1)^T\} + \sigma_v^2 I$$

Thus,

$$\underline{p} = E\{\underline{x}(n-1)\,x(n-1)^T\}\,\underline{a}$$

For $\underline{R}^{-1}\underline{p}$ to equal to $\underline{a}$ ~~to~~ must get rid of σ_v^2 term in $\underline{R}$

Since we ~~to~~ know σ_v^2 replace the $\underline{u}(n-1)\underline{u}(n-1)^T$ term ~~the $\underline{x}(n-1)\,\underline{x}(n-1)^T$~~ in the LMS with $\underline{u}(n-1)\,\underline{u}(n-1)^T - \sigma_v^2 I$. Thus

$$\underline{h}(n+1) = \underline{h}(n) + \mu\left(u(n)\,\underline{u}(n-1) - \left[\underline{u}(n-1)\,\underline{u}(n-1)^T - \sigma_v^2 I\right]\underline{h}(n)\right)$$

Limit (via independence assumption) satisfies

$$\underline{h} = \underline{h}(1 + \mu\sigma_v^2) + \mu(\underline{p} - \underline{R}\,\underline{h})$$

Question 2

(a) The main applications are Identification, Inverse modeling, Prediction and Interference cancellation. [For each application, a carefully label diagram with a sentence or two describing it will suffice. See lecture notes.]

(b) (i) RLS will adjust the filter weights $\underline{h} = [h_0, h_1, \ldots, h_{L-1}]^T$ where we assume L is known.

Let $e(k) = d(k) - \underline{h}^T \underline{u}(k)$

where $d(k)$ = desired response $= z(k) = y(k) + v(k)$

$$\underline{u}(k) = \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-L+1) \end{bmatrix}$$ (vector of current and L-1 previous inputs :)

The RLS cost function at time n is

$$J(\underline{h}, n) = \sum_{k=0}^{n} \lambda^{n-k} e(k)^2$$ wher

$\lambda \in (0, 1]$ is the forgetting factor.

$\lambda < 1$ implies past errors regarded as less important.

Let $\underline{h}(n) = \arg\min_{\underline{h}} J(\underline{h}, n)$. We can solve by differentiating the cost function and finding the zero of the derivative.

$$\frac{d}{d\underline{h}} J(\underline{h}, n) = \sum_{k=0}^{n} \lambda^{n-k} \, 2 e(k) \left(- \underline{u}(k) \right)$$

$$= \sum_{k=0}^{n} \lambda^{n-k} - 2 \left(d(k) - \underline{h}^T \underline{u}(k) \right) \underline{u}(k)$$

$$\sum_{k=0}^{n} \lambda^{n-k} \underline{u}(k) \underline{u}(k)^T \underline{h} = \sum_{k=0}^{n} \lambda^{n-k} d(k) \underline{u}(k)$$

$$\underline{h}(n) = \left(\sum_{k=0}^{n} \lambda^{n-k} \underline{u}(k) \underline{u}(k)^T \right)^{-1}$$

$$\times \sum_{k=0}^{n} \lambda^{n-k} d(k) \underline{u}(k)$$

Set $\lambda = 1$ and $\underline{h}(n) \rightarrow \underline{h}opt$ as $n \rightarrow \infty$

where $\underline{h}opt$ is the minimiser of $E\{e(k)^2\}$

The RLS algorithm computes the solution above recursively by using the Matrix Inversion Lemma:

$$\underline{h}(n) = \underline{h}(n-1) + S(n) \, \underline{u}(n) \left(d(n) - \underline{u}^T(n) \, \underline{h}(n-1) \right)$$

where $S(n)$ is a gain Matrix.

The LMS implements

$$\underline{h}(n) = \underline{h}(n-1) + \mu \, \underline{u}(n-1) \left(d(n-1) - \underline{h}^T(n-1) \underline{u}(n-1) \right)$$

Comparison metric	LMS	RLS
Comp. complexity	$O(L)$	$O(L^2)$
Stationary environment	$\underline{h}(n) \not\to \underline{h}_{opt}$	$\underline{h}(n) \to \underline{h}_{opt}$ for $\lambda = 1$
$E\{\underline{u}(k)\underline{u}(k)^T\}$ Sensitivity	high	Low

RLS is also quicker to converge.

(b)(ii) Note $\beta_0 = \beta_1 = 1$ (known).

So only have to estimate β_2.

$$d(k) = y(k) + v(k)$$

$$e(k) = d(k) - \beta_0 u(k) - \beta_1 u(k-1) - h_2 u(k-2)$$

Note we freeze $h_0 = \beta_0$ and $h_1 = \beta_1$

~~Can absorb~~

Let $\delta(k) = d(k) - \beta_0 u(k) - \beta_1 u(k-1)$

new $d(k)$

$$h_2(n) = \left[\sum_{k=0}^{n} \lambda^{n-k} u(k-2)^2 \right]^{-1} \times \sum_{k=0}^{n} \lambda^{n-k} \delta(k) u(k-2)$$

from previous part.

Q3 (a) Let $r_k = E\{x_n x_{n+k}\}$

$$x_n = a_2 x_{n-2} + b_0 w_n + b_1 w_{n-1}$$

square and take expectation:

$$r_0 = a_2^2 r_0 + \underbrace{b_0^2 + b_1^2}_{\text{call } \sigma^2}$$

multiply by x_{n-1} and take expectation:

$$r_1 = a_2 r_1 + b_1 b_0$$

multiply by x_{n-2} ...

$$r_2 = a_2 r_0$$

(b)(i) $p(x_2, x_4, \ldots, x_N \mid x_0)$

$$= p(x_N \mid x_{N-2}, \ldots, x_0) \times p(x_{N-2} \mid x_{N-4}, \ldots, x_0) \cdots \times p(x_2 \mid x_0)$$

$$= N(x_N, a_2 x_{N-2}, \sigma^2) \times N(x_{N-2}, a_2 x_{N-4}, \sigma^2) \cdots \times N(x_2, a_2 x_0, \sigma^2)$$

where $N(x, \mu, \sigma^2)$ is the ~~norm~~ normal $(\underset{\text{mean}}{\mu}, \underset{\text{variance}}{\sigma^2})$ density evaluated at x

(ii) take the log to get

$$\frac{N}{2} \log \frac{1}{\sqrt{2\pi}\,\sigma} - \frac{1}{2\sigma^2} \sum_{i=2}^{N} (x_i - a_2 x_{i-2})^2$$

even index only implicit

Solve for a_2 by usual least squares method:

$$\sum_{i=2}^{N} 2(x_i - a_2 x_{i-2})(-x_{i-2}) = 0$$

$$a_2^* = \frac{\sum_{i=2}^{N} x_i x_{i-2}}{\sum_{i=2}^{N} x_{i-2}^2}$$

Let ε = sum of squares ~~at~~ when $a_2 = a_2^*$

$$-\frac{N}{2} \times \frac{1}{2} \times \log \sigma^2 \quad - \frac{1}{2\sigma^2} \varepsilon$$

$$\frac{d}{d\sigma^2} = -\frac{N}{4}\frac{1}{\sigma^2} + \frac{\varepsilon}{2\sigma^4}$$

set to zero gives $\quad \sigma^2 = \dfrac{\varepsilon}{N/2}$

(c) From part (a)

$$r_0 = \frac{\sigma^2}{1 - a_2^2}$$

$$\cancel{x_n^2 = a_2 x_{n-2} x_n + (b_0 w_n + b_1 w_{n-1})(x(n)}$$

$$\cancel{\Rightarrow r_0 = a_2 r_2 + b_0^2 + b_1^2}$$

$$\left.\begin{aligned} b_1 b_0 &= r_1(1 - a_2) \\ \sigma^2 &= b_0^2 + b_1^2 \end{aligned}\right\} \text{can solve}$$

for b_0 and b_1

either directly or using spectral factorisation.

Question 4

(a) (i) $x_n = \cancel{\alpha} \alpha x_{n-2} + w_n \sigma$, $\sigma^2 = 1-\alpha^2$

where $\{w_n\}$ iid, zero mean, unit variance.

Let $r_0, r_1, \dots$ be the autocorrelation values

Check:

$$r_0 = \alpha^2 r_0 + \sigma^2$$

$$r_0 = \frac{\sigma^2}{1-\alpha^2} \Rightarrow \sigma^2 = 1-\alpha^2$$

$$r_1 = \alpha \cancel{x_n} r_1 \Rightarrow r_1 = 0$$

$$r_2 = \alpha r_0$$

$$r_3 = \alpha r_1 = 0$$

$$r_4 = \alpha r_2 = \alpha^2 r_0$$

$$\vdots$$

(a) (ii) $x_n = \sum_{i=0}^{9} w_{n-i}$, ~~w_n as in (a-i)~~

w_n iid, mean zero, variance 1

$r_0 = E(x_n^2) = 10$ as cross terms have zero expectation

$r_1 = E(x_{n+1} x_n) = 9$ as $x_{n+1} = w_{n-8} + w_{n-7} + \dots + w_n + w_{n+1}$

$x_n = w_{n-9} + w_{n-8} + \dots + w_n$

Each term of the product will have nine terms in common

$\vdots$

$r_{10} = E\{x_{n+10} x_n\} = 0$

(b)(i) $x_n = \alpha x_{n-1} + w_n$

$$r_0 = \alpha^2 r_0 + 1$$

$$r_1 = \alpha r_0$$

$$r_2 = \alpha r_1$$

$$\vdots$$

$$\Rightarrow r_k = \alpha^k r_0 , \quad r_0 = \frac{1}{1-\alpha^2}$$

Question 4

(b) (ii)

$$y_n = \beta y_{n-1} + x_n$$
$$= \beta(\beta y_{n-2} + x_{n-1}) + x_n$$
$$= \beta^2 y_{n-2} + \beta x_{n-1} + x_n$$
$$\vdots$$
$$y_n = \sum_{i=0}^{\infty} \beta^i x_{n-i}$$

$$y_{n+k} x_n = \sum_{i=0}^{\infty} \beta^i x_{n+k-i} x_n$$

$$E\{y_{n+k} x_n\} = \sum_{i=0}^{\infty} \beta^i R_{xx}[k-i]$$

$$R_{yx}[n+k, n] = R_{yx}[k]$$ (time shift dependence)

(b)(iii)

$$y_n = \beta y_{n-1} + x_n$$

$$y_n y_{n+k} = \beta y_{n-1} y_{n+k} + x_n y_{n+k}$$

$$R_{yy}[k] = \beta R_{yy}[k+1] + \underbrace{\sum_{i=0}^{\infty} \beta^i R_{xx}[k-i]}_{\text{convolution}}$$

$$S_y(e^{j\omega T}) = \beta S_y(e^{j\omega T}) e^{j\omega T} + \frac{1}{1-\beta e^{-j\omega T}} S_x(e^{j\omega T})$$

$$S_y = \frac{1}{1-\beta e^{j\omega T}} \frac{1}{1-\beta e^{-j\omega T}} S_x$$

$$= \frac{1}{|1-\beta e^{j\omega T}|^2} S_x = \frac{1}{(1-\beta\cos\omega T)^2 + \beta^2\sin^2\omega T} S_x$$

$$= \frac{S_x}{1+\beta^2-2\beta\cos\omega T}$$