Answers

2 (d) For $\omega << N$, $\mathbf{c}_g \to \frac{N}{k}\frac{\mathbf{k}_\perp}{k_\perp}$

For $\omega = N$, $\mathbf{c}_g \to 0$

3 (c) Condition is that κ is constant, not a function of position.

(d) Condition is $\mathbf{a} = 0$; functional is $J = \frac{1}{2}\int \nabla\phi \cdot \nabla\phi \, dV$

4 (a)(i) $\nabla^2 u + f = 0$; $\lambda + \alpha f = 0$; $\nabla^2 \lambda + u - u_{\text{obs}} = 0$

(a)(ii) Functional minimised is $J = \frac{1}{2}\int \left[(u - u_{\text{obs}})^2 + \alpha f^2 \right] dV$

(b)(ii) First two equations as in (a)(i);
plus $\nabla^2 \lambda + u - u_{\text{obs}} + \mu = 0$; $\mu \left[\int u \, dV - \beta V \right] = 0$ (can't localise this one)

ENGINEERING TRIPOS IIA/IIB 2013 MODULE 4M12
SOLUTIONS

1 (i) $$T(x,t) = \frac{1}{2\sqrt{\pi \alpha t}} \exp[-x^2/4\alpha t]$$

$$\frac{\partial T}{\partial t} = -\frac{1}{2}\frac{T}{t} + \frac{x^2}{4\alpha t^2} T$$

$$\frac{\partial T}{\partial x} = -\frac{x}{2\alpha t} T \Rightarrow \frac{\partial^2 T}{\partial x^2} = -\frac{1}{2\alpha t} T + \frac{x^2}{4\alpha^2 t^2} T$$

$$\Rightarrow \alpha \frac{\partial^2 T}{\partial x^2} = -\frac{1}{2t} T + \frac{x^2}{4\alpha^2 t} T = \frac{\partial T}{\partial t} \quad \checkmark$$

For $x = 0$, $T(0, t \to 0) \to \infty$

For $x \neq 0$, $T(x, t \to 0) \sim \frac{1}{\sqrt{t}} \exp\left[-\frac{1}{t}\right] \to 0$

$$\int_{-\infty}^{\infty} T\,dx = \frac{1}{2\sqrt{\pi \alpha t}} \int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{4\alpha t}\right] dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz = 1$$

$$(z = x/2\sqrt{\alpha t})$$

So $T(t \to 0)$ is a δ-function. (35%)

(ii) Solution for δ-function of strength T_0 located at $x = x'$

$$T(x,t) = \frac{T_0}{2\sqrt{\pi \alpha t}} \exp\left[-\frac{(x-x')^2}{4\alpha t}\right]$$

By super-position, solution to initial condition $T_0(x')$ is then

$$T(x,t) = \int_{-\infty}^{\infty} \frac{T_0(x')}{2\sqrt{\pi \alpha t}} \exp\left[\frac{-(x-x')^2}{4\alpha t}\right] dx'$$ (30%)

(iii) $$T(x,t) = \hat{T} \int_{0}^{\infty} \frac{1}{2\sqrt{\pi \alpha t}} \exp\left[-\frac{(x-x')^2}{4\alpha t}\right] dx'$$

Let $z = x/2\sqrt{\alpha t}$

$$T(x,t) = \frac{\hat{T}}{\sqrt{\pi}} \int_0^\infty e^{-(z-z')^2}\, dz'$$

$$= \frac{\hat{T}}{\sqrt{\pi}} \int_{-z}^\infty e^{-p^2}\, dp \qquad p = z' - z$$

$$= \frac{\hat{T}}{\sqrt{\pi}} \int_0^\infty e^{-p^2}\, dp + \frac{\hat{T}}{\sqrt{\pi}} \int_0^z e^{-p^2}\, dp$$

$$= \frac{\hat{T}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} + \frac{\hat{T}}{2} \operatorname{erf} z$$

$$= \frac{1}{2}\hat{T}\left[1 + \operatorname{erf} z\right]$$

$$= \underline{\underline{\frac{1}{2}\hat{T}\left[1 + \operatorname{erf}(x/2\sqrt{\alpha t})\right]}} \qquad (35\%)$$

2 (i) $\eta = A_0 \cos(k_1 x - \omega_1 t) + A_0 \cos(k_2 x - \omega_2 t)$

$= A_0\, 2 \cos\left(\tfrac{1}{2}(k_1+k_2)x - \tfrac{1}{2}(\omega_1+\omega_2)t\right) \cos(\delta k x - \delta\omega t)$

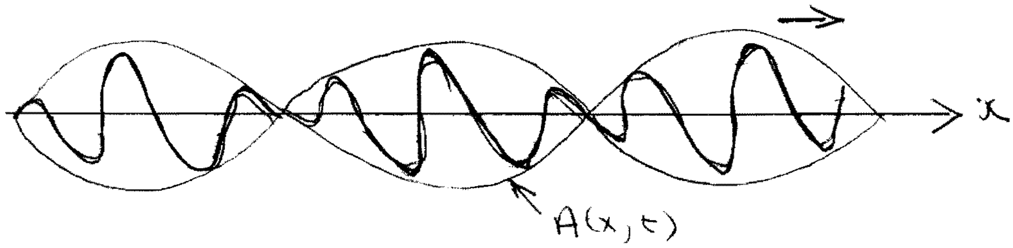
$= 2A_0 \cos(\bar{k}x - \bar{\omega}t) \cos(\delta k\, x - \delta\omega t)$

$\bar{k} = \tfrac{1}{2}(k_1+k_2)\quad , \quad \bar{\omega} = \tfrac{1}{2}(\omega_1+\omega_2)$

Rewrite as

$$\eta = A(x,t) \cos(\bar{k}x - \bar{\omega}t)$$

$$A = 2\cos(\delta k x - \delta\omega t) = 2\cos(\delta k(x - c_g t))$$



Amplitude propagates in the x-direction at speed

$$c_g = \delta\omega / \delta k \qquad (20\%)$$

(ii) $\frac{\partial^2}{\partial t^2} \nabla^2 u_z + N^2 \nabla_\perp^2 u_z = 0$

$\Rightarrow \quad (-\omega^2)(-k^2)\hat{u}_z + N^2(-k_\perp^2)\hat{u}_z = 0$

$\Rightarrow \quad \omega^2 = \frac{N^2 k_\perp^2}{k^2}$

$\Rightarrow \quad \underline{\underline{\omega = N k_\perp / k}}$

$$c_{g,x} = \frac{\partial \omega}{\partial k_x} = \frac{N}{k}\frac{\partial k_\perp}{\partial k_x} + N k_\perp \frac{\partial}{\partial k_x}\left(\frac{1}{k}\right)$$

$$= \frac{N}{k}\frac{k_x}{k_\perp} + N k_\perp\left(-\frac{k_x}{k^3}\right)$$

$$= \frac{N k_x}{k^3 k_\perp}\left(k^2 - k_\perp^2\right)$$

$$= \frac{N k_x k_z^2}{k^3 k_\perp} \quad \text{(similar for } c_{g,y}\text{)}$$

$$c_{g,z} = \frac{\partial \omega}{\partial k_z} = N k_\perp \frac{\partial}{\partial k_z}\left(\frac{1}{k}\right) = N k_\perp\left(-\frac{k_z}{k^3}\right) = -\frac{N k_z k_\perp^2}{k^3 k_\perp}$$

$$\Rightarrow \underline{c}_g = \frac{N}{k^3 k_\perp}\left(k_x k_z^2,\; k_y k_z^2,\; -k_z k_\perp^2\right)$$

Compare,

$$\underline{c}_g = \frac{N}{k^3 k_\perp}\left(k_\parallel^2 \underline{k}_\perp - k_\perp^2 \underline{k}_\parallel\right) = \frac{N}{k^3 k_\perp}\left(k_z^2 k_x,\; k_z^2 k_y,\; -k_\perp^2 k_z\right)$$

✓

Phase velocity is $\parallel$'el to $\underline{k}$,

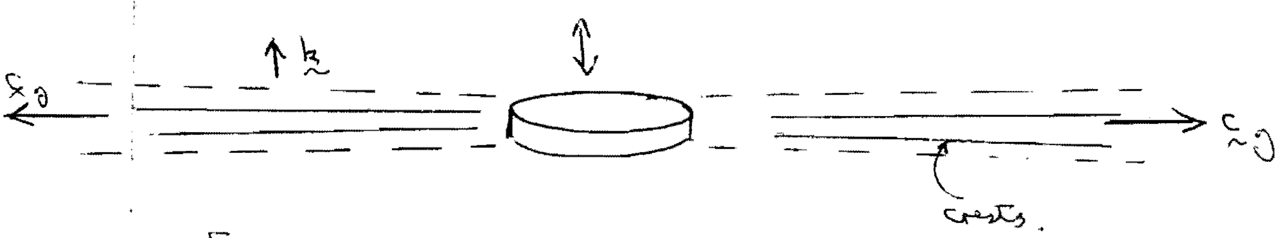
$$\underline{c}_g \cdot \underline{k} = \frac{N}{k^3 k_\perp}\left(k_\parallel^2 \underbrace{\underline{k}_\perp \cdot \underline{k}}_{k_\perp^2} - k_\perp^2 \underbrace{\underline{k}_\parallel \cdot \underline{k}}_{k_\parallel^2}\right) = 0$$

(45%)

(iii)

For $\omega \to 0$, $k_\perp \to 0$ and $k \to k_z$

$$\Rightarrow \underline{k} \approx k_\parallel \hat{\underline{e}}_z, \quad \underline{c}_g \approx \frac{N \underline{k}_\perp}{k\, k_\perp} = \frac{N}{k} \cdot \frac{\underline{k}_\perp}{k_\perp}$$

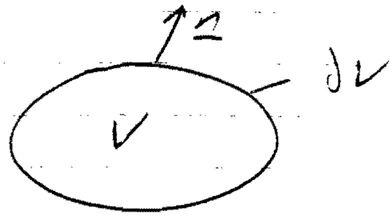


For $\omega \to N$, $k_z \to 0$ $\Rightarrow$ $\underline{c}_g = 0$, No dispersion of energy

(35%)

3.

a)



Conservation of ϕ:

$$\frac{d}{dt}\int_V \phi \, dV = \underbrace{-\int_{\partial V} (\phi \underline{a}) \cdot \underline{n} \, dS}$$

rate of increase of ϕ in V

rate at which ϕ enters domain (-ve sign because $\underline{n}$ points outward)

Apply divergence theorem:

$$\frac{d}{dt}\int_V \phi \, dV = -\int_V \nabla \cdot (\phi \underline{a}) \, dV$$

Must also hold for an arbitrary sub-volume $V_s \subset V$, hence can localise:

$$\frac{d\phi}{dt} + \nabla \cdot (\phi \underline{a}) = 0$$

- Expand, $d\phi/dt + \nabla\phi \cdot \underline{a} + \phi \nabla \cdot \underline{a} = 0$

Since $\nabla \cdot \underline{a} = 0$ (incompressible),

$$d\phi/dt + \underline{a} \cdot \nabla\phi = 0 \, . \quad (30\%)$$

3b. $d\phi/dt + a_i \dfrac{\partial \phi}{\partial x_i} = 0$ (10%)

c) $\underset{\sim}{q} = \phi \underline{a} - k \nabla \phi$

Balance:

$$\frac{d}{dt}\int_V \phi \, dV = -\int_{\partial V} (\phi \underline{a} + k\nabla\phi)\cdot\underline{n} \, dS$$

Diverg. theorem:

$$\frac{d}{dt}\int \phi \, dV = -\int \nabla\cdot(\phi a) + \nabla\cdot(k\nabla\phi) \, dS$$

$$\rightarrow \frac{d\phi}{dt} + \nabla\cdot(\phi \underline{a}) - \nabla\cdot(k\nabla\phi) = 0$$

Expand

$$\frac{d\phi}{dt} + \nabla\phi\cdot\underline{a} + \phi \cancel{\nabla\cdot\underline{a}}^{\,0} - \nabla k\cdot\nabla\phi - k\nabla^2\phi = 0$$

$\longrightarrow$ k must be constant. (30%)

d) Steady state $\underline{a}\cdot\nabla\phi - k\nabla^2\phi = 0$

Multiply by w & integrate

$$a(w,u) = \int_V w \, \underline{a}\cdot\nabla\phi \, dV - \int w k \nabla^2\phi \, dV = 0$$

Int by parts last term:

$$a(w,u) = \int_V w \, \underline{a}\cdot\nabla\phi \, dV + \int_V \nabla w\cdot(k\nabla\phi) - \underline{\int_{\partial V} w k \nabla\phi\cdot n \, dS}$$

d) cont. Bilinear form a is symmetric
when $\underline{a} = \underline{0}$

→ Corresponds to Poisson problem:

$$J = \frac{1}{2}\int \nabla\phi \cdot \nabla\phi \, dV \cancel{- \int f\phi}$$ (30%)

4

a) i) Unknown are λ, u & f.
Take directional derivative (variation) w.r.t. each unknown & set equal to zero.

$$DF_{\bar{\lambda}} = \int_V \bar{\lambda}(\nabla^2 u + f)\, dV = 0 \quad (1)$$

$$DF_{\bar{u}} = \int_V \lambda(\nabla^2 \bar{u}) + (u - u_{OBS})\bar{u}\, dV = 0 \quad (2)$$

$$DF_{\bar{f}} = \int_V \lambda \bar{f} + \alpha f \bar{f}\, dV = 0 \quad (3)$$

Follow usual process to ~~isolat~~ isolate $\bar{\cdot}$,

(1) $\nabla^2 u + f = 0$

(2) Integrate by parts twice

$$-\int_V \nabla\lambda \cdot \nabla\bar{u} + (u - u_{OBS})\bar{u}\, dV + \int_{\partial V} \lambda \nabla\bar{u}\cdot n\, dS$$

Again $\int \nabla^2\lambda\, \bar{u} + (u - u_{OBS})\bar{u}\, dV$ $+ \int \lambda \nabla\bar{u}\cdot n\, dS$ $- \int \nabla\lambda\cdot n\, \bar{u}\, dS$ 0 from bcs.

$\longrightarrow \nabla^2\lambda + (u - u_{OBS}) = 0$

(3) $\lambda + \alpha f = 0$ (30%)

ii) $$J = \int_V \tfrac{1}{2}(u - u_{OBS})^2 + \tfrac{\alpha}{2} f^2\, dV$$ (10%)

9

b) i) Need to add extra constraint: $\frac{1}{V}\int_V u\,dV - \beta = 0$

$\rightarrow$ add extra Lagrange multiplier:

$$F^{mod} = \int_V \lambda_1 (\nabla^2 u + f) + \frac{1}{2}(u - u_{obs})^2 + \frac{\alpha}{2} f^2$$

$$+ \lambda_2 \left[\int_V u\,dV - \beta V\right]$$

does not depend on position

(20%)

ii) Compute directional derivative w.r.t λ_1, λ_2, u & f

$$DF_{\bar\lambda_1} = \int \bar\lambda_1 (\nabla^2 u + f)\,dV = 0$$

$$\longrightarrow \quad \nabla^2 u + f = 0$$

$$DF_{\bar\lambda_2} = \bar\lambda_2\left[\int u\,dV - \beta V\right] = 0$$

(cannot localize this expression)

$$DF_{\bar u} = \int \lambda_1 (\nabla^2 \bar u \cancel{+f}) + (u - u_{obs})\bar u\,dV$$

$$+ \lambda_2 \int \bar u\,dV = 0$$

$$\rightarrow \int \nabla^2 \lambda_1 \bar u + (u - u_o)\bar u + \lambda_2 \bar u\,dV = 0$$

$$\longrightarrow \quad \nabla^2 \lambda_1 + u - u_{obs} + \lambda_2 = 0$$

$$DF_{\bar f} = \int_V \lambda_1 \bar f + \alpha f \bar f\,dV$$

$$\longrightarrow \quad \lambda_1 + \alpha f = 0 \qquad (40\%)$$

Note: ~~Method proper~~ To solve numerically,

[illegible] weak [illegible] Galerkin method set