ENGINEERING TRIPOS PART IIB

Friday 3 May 2013 9.30 to 11

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: 4A12 Data Card

(i) Vortex Dynamics (1 page)

(ii) Turbulence (2 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

EM3

1 (a) State Helmholtz's two laws of inviscid vortex dynamics. Provide a simple proof of the first of these laws by noting that a short line element in the fluid, $\delta \mathbf{r}$, which always consists of the same fluid particles, evolves according to

$$\frac{D}{Dt}(\delta \mathbf{r}) = (\delta \mathbf{r} \cdot \nabla)\mathbf{u}$$

[20%]

(b) State Kelvin's theorem for an inviscid fluid. An alternative proof of Helmholtz's first law can be constructed using Kelvin's theorem. Consider a thin isolated vortex tube and a closed curve C that encircles it at $t = 0$. Use Kelvin's theorem to show that, if C is a material curve that is always composed of the same fluid particles, then C must encircle the vortex tube for all time. Use this fact to deduce Helmholtz's first law. [20%]

(c) A vorticity field, $\boldsymbol{\omega}(\mathbf{x},t)$, consists of two, thin, closed vortex tubes whose vorticity fluxes are Φ_1 and Φ_2 and whose centrelines are C_1 and C_2. The net helicity of the flow associated with the tubes is defined as $H = \int \mathbf{u} \cdot \boldsymbol{\omega} dV$.

(i) Confirm that

$$H = \oint_{C_1} \mathbf{u} \cdot (\Phi_1 d\mathbf{r}) + \oint_{C_2} \mathbf{u} \cdot (\Phi_2 d\mathbf{r})$$

[20%]

(ii) Use Stokes' theorem to show that $H = 0$ if the vortex tubes are not linked, but that $H = \pm 2\Phi_1\Phi_2$ if the tubes are interlinked. [20%]

(d) It may be shown that H is conserved provided the flow is inviscid. Explain this using Helmholtz's laws. [20%]

2 (a) What three physical processes must balance in order to obtain a steady solution of the vorticity equation below?

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}$$ [20%]

(b) Consider the vortex sheet

$$\boldsymbol{\omega} = \frac{\Phi}{\sqrt{\pi}\, w} \exp\left(-\frac{z^2}{w^2}\right)\hat{\mathbf{e}}_y$$

where Φ is a constant, z is the axial coordinate, $\hat{\mathbf{e}}_y$ is a unit vector in the y-direction, and w is the characteristic thickness of the sheet. The velocity field associated with this vortex sheet is $\mathbf{u}_{VS} = u_x(z)\hat{\mathbf{e}}_x$.

(i) Show that Φ is the flux of vorticity along the sheet per unit length of the x-direction. You will need to make use of the definite integral $\int_{-\infty}^{\infty} \exp[-s^2]ds = \sqrt{\pi}$. [20%]

(ii) The vortex sheet is placed in an externally imposed, irrotational velocity field $\mathbf{u}_{SF} = (0, \alpha y, -\alpha z)$ in (x, y, z) coordinates, where α is a constant. Show that, if $\alpha = 2\nu/w^2$, where ν is the kinematic viscosity, this constitutes an exact, steady solution of the vorticity equation. [40%]

(iii) What will happen to the vortex sheet with time if, at $t = 0$, the externally applied strain field satisfies $\alpha > 2\nu/w^2$ and what if $\alpha < 2\nu/w^2$? [20%]

3 Consider an air flow at ambient conditions with stationary homogeneous isotropic non-decaying turbulence with integral length scale $L = 0.1\,\mathrm{m}$, characteristic turbulent velocity fluctuation $u = 1\,\mathrm{ms}^{-1}$, and mean velocity in the streamwise direction $U = 10\,\mathrm{ms}^{-1}$. There is no mean flow in the other directions. A hot wire is placed in the flow.

(a) Estimate the turbulent Reynolds number, the Reynolds number based on the Taylor scale, and the Kolmogorov length and timescales. [40%]

(b) Provide an estimate of the fastest frequency that the hot wire must follow and of the integral timescale of the velocity time series detected by the hot wire. [30%]

(c) Discuss what is meant by *Taylor's hypothesis* and whether it is valid for this flow. [30%]

4 Consider the stagnation region between two opposed turbulent flows. With the coordinate system fixed at the stagnation point, the mean velocities are completely characterized by $U_1 = Ax_1$, $U_2 = -Ax_2$, and $U_3 = 0$, where A is a positive constant. (This is also called Hiemenz flow.) The production of turbulent stress $\overline{u_i u_j}$ is given by P_{ij}, where

$$P_{ij} = -\left(\overline{u_i u_k} \frac{\partial U_j}{\partial x_k} + \overline{u_j u_k} \frac{\partial U_i}{\partial x_k} \right)$$

with summation implied over the index k. The production of the turbulent kinetic energy is given by

$$P = -\overline{u_i u_j} \frac{\partial U_i}{\partial x_j}$$

with repeated summation implied over indices i and j.

(a) By considering separately the production for each normal stress, show that $P_{11} = -2A\overline{u_1^2}$, $P_{22} = 2A\overline{u_2^2}$, and $P_{33} = 0$. [40%]

(b) Show that the production of kinetic energy is $P = A\left(\overline{u_2^2} - \overline{u_1^2}\right)$. [30%]

(c) Compare the result from Part (b) with the production term in the k-ε turbulence model given in the Data Card as applied to this flow. Are they consistent? If not, why not? [30%]

END OF PAPER

Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint A \cdot d\boldsymbol{S}$

Stokes : $\int_S (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

Vector Identities

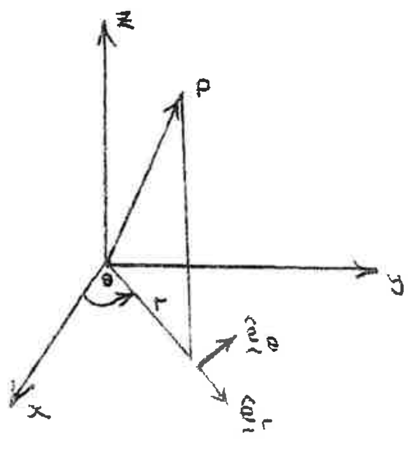
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot A) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \right)$$

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4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$)**:**

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ &\quad - \overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}\end{aligned}$$

The $k - \varepsilon$ model:

$$\frac{\partial k}{\partial t} + \overline{u}_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + P_k - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k}P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

$$P_k = \frac{1}{2}\nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)^2$$

$$C_\mu = 0.09, \;\; c_{\varepsilon 1} = 1.44, \;\; c_{\varepsilon 2} = 1.92, \;\; \sigma_k = 1.0, \;\; \sigma_\varepsilon = 1.3$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi'\phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D\frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'}\frac{\partial \overline{\phi}}{\partial x_j} - 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Scalar fluctuations (modelled):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_i \frac{\partial \sigma^2}{\partial x_i} = \frac{\partial}{\partial x_i}\left((D + D_{turb})\frac{\partial \sigma^2}{\partial x_i}\right) + 2D_{turb}\left(\frac{\partial \overline{\phi}}{\partial x_i}\right)^2 - 2\overline{N}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2 = 2\frac{u}{L_{turb}}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned} \eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\ \tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\ v_K &= \left(\nu\varepsilon\right)^{1/4} \end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned} \overline{u_i'u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\ \overline{u_j'\phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j} \end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1'u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$