ENGINEERING TRIPOS PART IIB

Friday 3 May 2013 9.30 to 11

Module 4F3

OPTIMAL AND PREDICTIVE CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Data sheet (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

kg03

1 (a) (i) State the definition of a *convex function* $f(z)$, where $z \in \mathbb{R}^n$ and $f(z) \in \mathbb{R}$. [10%]

(ii) For $f(x,y) = x^2 + y^2 + 4xy$, show that this is a convex function of x for any fixed value of y and a convex function of y for any fixed value of x, but is not a convex function of (x,y). [15%]

(b) A linear plant with state x_k at time k and input u_k at time k is described by the discrete-time state space model: $x_{k+1} = Ax_k + Bu_k$.

(i) Let $\underline{\theta} \triangleq [u_0^T, x_1^T, u_1^T, x_2^T, u_2^T, x_3^T]^T$ be a vector of interleaved predicted future inputs and states. For compatibly sized matrices Q, R and P, and a given initial state x_0, an unconstrained predictive controller minimises the finite horizon cost function

$$V(x_0, \underline{\theta}) = x_3^T P x_3 + \sum_{k=0}^{2} \left(x_k^T Q x_k + u_k^T R u_k \right).$$

When the states x_i $(i = 1, \ldots, 3)$ are **not** eliminated algebraically, this can be written as a mathematical optimisation problem in the standard form:

$$\begin{aligned} \text{Minimize:} \quad & \underline{\theta}^T H \underline{\theta} \\ \text{Subject to:} \quad & F\underline{\theta} - f = 0. \end{aligned}$$

Find the matrices H and F, and the vector f. [30%]

(ii) State sufficient conditions on Q, R and P for the problem to be convex with respect to $\underline{\theta}$ and have a unique solution. [10%]

(iii) The plant has a single output, i.e. $y_k = Cx_k$ is scalar. The controller from part (b)(ii) is modified to constrain the output y_k to lie in the range $-y_{\max} \leq y_k \leq y_{\max}$ for $k = 1, \ldots, 3$, where $y_{\max} > 0$. State the modified constrained convex optimisation problem in standard form in terms of H, F, f, C, $y_{\max}$ and $\underline{\theta}$. [20%]

(iv) The controller is modified so that each element of u_i may only take values in the range $[-u_{\max}, -\epsilon] \cup 0 \cup [\epsilon, u_{\max}]$ (i.e. the inputs may be zero, **or**, their magnitudes must be between ϵ and $u_{\max}$) for $u_{\max} > \epsilon > 0$. Comment on the practical implications of this modification. [15%]

2 (a) Predictive controllers usually employ a fixed but receding prediction horizon.

(i) Explain the principle of the receding horizon, paying attention to why control using a receding horizon controller based on open-loop predictions is a closed-loop feedback control law. [20%]

(ii) State some merits and drawbacks of predictive control. [20%]

(b) Consider the state space system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + B_d d_k \\ y_k &= [x_k^T, d_k^T]^T \\ z_k &= Hx_k \end{aligned}$$

where the vector x_k is the state at time k, u_k is the input, y_k is the measured output, z_k is the controlled output, $d_k = d$ is a constant disturbance and the pair (A, B) is reachable. An unconstrained predictive controller with the cost function

$$V(x_0, \mathbf{u}) = (x_N - x_s)^T P(x_N - x_s) + \sum_{k=0}^{N-1} \left[(x_k - x_s)^T Q(x_k - x_s) + (u_k - u_s)^T R(u_k - u_s) \right]$$

is designed where $x_s = 0$, $u_s = 0$, $\mathbf{u} = \left[u_0^T \; u_1^T \; \cdots \; u_{N-1}^T \right]^T$, $Q > 0$ and $R > 0$. The spectral radius $\rho(A) < 1$, and $A^T PA - P + Q = 0$. The prediction model exactly matches the plant (including a perfect model of the disturbance) and measurements are free of noise and bias. Nevertheless, the design is still unsatisfactory since the output z_k does not converge to zero.

(i) What in the problem formulation is preventing convergence of z to zero? [15%]

(ii) Without changing the form of the cost function, and considering that the numerical values of Q, R and P may be changed by a third party at some point in the future, how should the setpoints x_s and u_s be modified so that the controller tracks zero without offset? [20%]

(iii) Suppose the controller must now drive z to track a reference r without offset. What conditions should the new pair (x_s, u_s) satisfy? [5%]

(iv) What conditions should A, B, and H satisfy to guarantee that a suitable pair (x_s, u_s) exists for any compatibly dimensioned d and r? How many control inputs are needed to meet these conditions? [20%]

3 (a) Explain what is meant by a *balanced realisation* of a matrix transfer function, $G(s) = C(sI - A)^{-1} B$, and comment on its possible use in producing reduced order approximate state-space models. [35%]

(b) Consider the system described by the state equation:

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & 0 & B_2 \\ C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ C_2 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x \\ w_1 \\ w_2 \\ u \end{bmatrix}$$

where x is the state, w_1 and w_2 are disturbances, u is the controller output, z_1, z_2 and y are system outputs, and

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(i) It is desired to design a state feedback controller to minimize the $\mathcal{H}_2$ norm of the transfer function from w_1 to $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. Verify that the Control Algebraic Riccati Equation (CARE) is solved by $X = \begin{bmatrix} \pm 1 & 1 \\ 1 & \pm 2 \end{bmatrix}$, and determine which solution solves the control problem. Calculate the feedback gain and the corresponding closed-loop poles. [35%]

(ii) Now suppose that the signal y rather than x is measured and available to the controller and it is desired to minimise the $\mathcal{H}_2$ norm of the transfer function from $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to z. Comment on what would result from using the standard equations for the controller as given in the data sheet with $Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. In particular comment on the resulting closed-loop poles. [30%]

4 (a) Briefly discuss the use of the $\mathcal{H}_\infty$-norm to assess system performance. For a system with transfer function $G(s) = C\,(sI - A)^{-1}\,B$ outline two methods for calculating $\|G(s)\|_\infty$. [35%]

(b) Consider the block diagram in Fig. 1.

(i) Determine the transfer function $P(s)$ such that

$$\bar{z}(s) = \mathcal{F}_\ell\,(P(s), K(s))\,\bar{w}(s)$$

where $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. [20%]

(ii) Given the state-space realisations $G_a(s) = C_a\,(sI - A_a)^{-1}\,B_a$ and $G_b(s) = C_b\,(sI - A_b)^{-1}\,B_b$, find a state-space realisation for $P(s)$. [20%]

(iii) Describe the procedure to find a controller $K(s)$ that will minimise $\|\mathcal{F}_\ell\,(P(s), K(s))\|_\infty$. [25%]

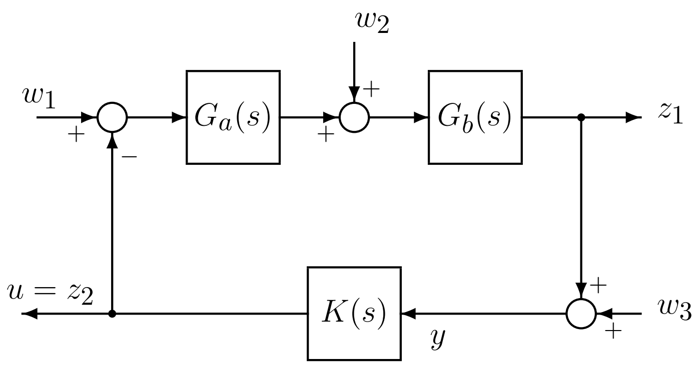

Fig. 1

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