ENGINEERING TRIPOS PART IIB

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Tuesday 30 April 2013 2 to 3.30

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Module 4F7

# DIGITAL FILTERS AND SPECTRUM ESTIMATION

*Answer not more than* ***three*** *questions.*

*All questions carry the same number of marks.*

*The* ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

*There are no attachments.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

1 Let

$$x(n) = \sum_{i=1}^{L} a_i x(n-i) + w(n)$$

where $\{w(n)\}$ is an independent and identically distributed zero-mean unit-variance noise sequence.

(a) Formulate the Wiener filtering problem whose solution is $(a_1, a_2, \ldots, a_L)$. [20%]

(b) Write down the Steepest Descent algorithm for estimating $(a_1, a_2, \ldots, a_L)$. [15%]

(c) Write down the Least Mean Square (LMS) algorithm for estimating $(a_1, a_2, \ldots, a_L)$. [10%]

(d) Rather than $x(n)$ itself, only noisy observations $u(n) = x(n) + v(n)$ are available where $\{v(n)\}$ is an independent and identically distributed zero-mean noise sequence, independent of $\{w(n)\}$, with variance $\sigma_v^2$.

(i) Write down the LMS algorithm for estimating $(a_1, a_2, \ldots, a_L)$ and characterise the limit of the expected value of the LMS estimates. [35%]

(ii) Assuming knowledge of $\sigma_v^2$, modify your LMS algorithm so that the expected value of the LMS estimates converge to $(a_1, a_2, \ldots, a_L)$. You must also prove convergence. (Hint: In your LMS algorithm, use an estimate of the autocorrelation of $\{x(n)\}$ instead.) [20%]

2 (a) Describe, with the definition of the relevant signals, the four main applications of adaptive filters. [30%]

(b) Figure 1 depicts a finite impulse response (FIR) identification problem. The coefficients of the FIR filter, $(\beta_0, \beta_1, \ldots, \beta_{L-1})$, are to be estimated using an input sequence $\{u(n)\}_{n \geq 0}$. However, only noisy measurements $z(n) = y(n) + v(n)$ of the FIR filter output $y(n)$ are available where $\{v(n)\}$ is an independent and identically distributed zero-mean noise sequence with variance $\sigma_v^2$.

(i) Describe in detail the Recursive Least Squares (RLS) method for adaptively estimating the coefficients of the FIR filter and contrast its performance with the Least Mean Square method. [40%]

(ii) Let $L = 3$ and assume knowledge of $\beta_0 = \beta_1 = 1$. Compute the RLS estimate of $\beta_2$ at time $n$. State the limit of the RLS estimate. [30%]

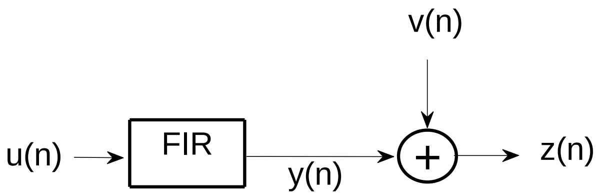


Fig. 1

3 Consider the following ARMA(2,2) model

$$x_n = a_2 x_{n-2} + b_0 w_n + b_1 w_{n-1}$$

where $\{w_n\}$ is a sequence of independent Gaussian random variables with zero mean and unit variance. Let $R_{XX}[k] = E\{x_{n+k}x_n\}$.

(a) Derive 3 simultaneous equations for $R_{XX}[0]$, $R_{XX}[1]$ and $R_{XX}[2]$. [25%]

(b) Consider the even index variables $(x_0, x_2, \ldots, x_N)$ where $N \geq 2$ is even.

(i) Write down the expression for the joint probability density of $(x_2, x_4, \ldots, x_N)$ given $x_0$. [20%]

(ii) Solve for the Maximum Likelihood estimates of $a_2$ and $b_0^2 + b_1^2$ using just $(x_0, x_2, \ldots, x_N)$. [35%]

(c) You are given $(x_0, x_1, x_2, \ldots, x_N)$ from which you form an estimate of $R_{XX}[1]$. Describe in detail how estimates for $b_0$ and $b_1$ may now be obtained. [20%]

4 (a) For a scalar random process $\{x_n\}$, let $R_{XX}[k] = E\{x_{n+k}x_n\}$. For each autocorrelation function $R_{XX}$ below, give the equation of the corresponding ARMA process and verify your answer.

(i) $R_{XX}[2k] = \alpha^k$ and $R_{XX}[2k+1] = 0$ where $|\alpha| < 1$. [20%]

(ii) $R_{XX}[k] = 10 - |k|$ for $|k| \leq 10$, and $R_{XX}[k] = 0$ otherwise. [20%]

(b) Let $y_n = \beta y_{n-1} + x_n$, $|\beta| < 1$, where $x_n$ is the AR(1) process

$$x_n = \alpha x_{n-1} + w_n$$

with $|\alpha| < 1$, and $w_n$ is a sequence of independent random variables with zero mean and unit variance.

(i) Obtain the expression for $R_{XX}[k]$. [15%]

(ii) Write $y_n$ as a convolution of the sequence $\{x_k\}$ and then derive the expression for $R_{YX}[n+k,n] = E\{y_{n+k}x_n\}$. [15%]

(iii) Obtain the power spectrum of $\{y_n\}$ in terms of the power spectrum of $\{x_n\}$. (Hint: For example, first obtain an expression for $R_{YY}[k]$ in terms of $R_{YY}[k+1]$ and $R_{YX}[n+k,n]$.) [30%]

**END OF PAPER**