ENGINEERING TRIPOS PART IIA
ENGINEERING TRIPOS PART IIB

Monday 22 April 2013 2 to 3.30

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: 4M12 Data Sheet (2 sides).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

jw03

1 (a) Show that one solution of the one-dimensional heat diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2},$$

is

$$T(x,t) = \frac{1}{2\sqrt{\pi \alpha t}} \exp\left[-x^2/4\alpha t\right], \quad -\infty < x < \infty,$$

where T is the temperature, x the spatial coordinate, and α the thermal diffusivity. Confirm that the initial condition at $t = 0$ corresponding to this particular solution is the delta-function $\delta(x)$. You may find it useful to note that

$$\int_{-\infty}^{\infty} \exp\left(-y^2\right) dy = \sqrt{\pi}\,. \qquad [35\%]$$

(b) Now use superposition to show that the general solution in the domain $-\infty < x < \infty$ corresponding to the initial condition $T\left(x, t=0\right) = T_0(x)$ is

$$T(x,t) = \frac{1}{2\sqrt{\pi \alpha t}} \int_{-\infty}^{\infty} T_0(x') \exp\left[\frac{-(x-x')^2}{4\alpha t}\right] dx'\,. \qquad [30\%]$$

(c) Hence, or otherwise, show that the temperature distribution for $x > 0$ corresponding to the initial condition

$$T_0(x) = \begin{cases} 0 & x < 0 \\ \hat{T} & x \geq 0 \end{cases}$$

where $\hat{T}$ is a constant, is given by

$$T(x,t) = \frac{\hat{T}}{2}\left[1 + \operatorname{erf}\left(x/\sqrt{4\alpha t}\right)\right],$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2)\,dz$ is the usual error function. [35%]

2 (a) Consider the superposition of two waves of similar wavenumber and frequency:

$$\eta(x,t) = A_0 \cos\left(k_1 x - \omega_1 t\right) + A_0 \cos\left(k_2 x - \omega_2 t\right),$$

where $k_2 = k_1 + \delta k$ and $\omega_2 = \omega_1 + \delta\omega$. Show that this takes the form of a slowly modulated wave-train whose envelope propagates at the group speed

$$c_g = \frac{\delta\omega}{\delta k} \approx \frac{d\omega}{dk}.$$

[20%]

(b) Internal gravity waves in a stratified fluid are governed by

$$\frac{\partial^2}{\partial t^2}\left(\nabla^2 u_z\right) + N^2 \nabla_\perp^2 u_z = 0,$$

where u_z is the vertical velocity, N is a constant which measures the strength of the stratification, and $\nabla_\perp^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal contribution to the Laplacian. Show that the dispersion relationship corresponding to plane waves $u_z = \hat{u}_z \exp\left[j\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right]$ is

$$\omega = N k_\perp / |\mathbf{k}|, \quad \text{where } k_\perp = \sqrt{k_x^2 + k_y^2} \text{ and } \mathbf{k} = \left[k_x, k_y, k_z\right]^T.$$

Hence show that the group velocity of internal gravity waves is given by

$$\mathbf{c}_g = \frac{N}{|\mathbf{k}|^3 k_\perp}\left[k_{//}^2 \mathbf{k}_\perp - k_\perp^2 \mathbf{k}_{//}\right],$$

where $\mathbf{k}_{//} = k_z \hat{\mathbf{e}}_z$ and $\mathbf{k}_\perp = \mathbf{k} - \mathbf{k}_{//}$. Confirm that $\mathbf{c}_g$ is perpendicular to the phase velocity. [45%]

(c) A horizontal disc oscillates vertically in an unbounded, stratified fluid as in part (b), with angular frequency ω. Calculate the group velocity and sketch the wave patterns corresponding to the two frequencies $\omega << N$ and $\omega = N$. In each case show the orientation of the wave crests relative to $\mathbf{c}_g$. [35%]

3 The flux of a conserved quantity is given by $\mathbf{q} = \mathbf{a}\phi$ where ϕ is the concentration of the conserved scalar quantity and $\mathbf{a}$ is an incompressible velocity field.

(a) By considering conservation of ϕ for a domain V with boundary ∂V, show that the governing partial differential equation for advective transport is

$$\frac{\partial \phi}{\partial t} + \mathbf{a} \cdot \nabla \phi = 0 \quad . \tag{1}$$

[30%]

(b) Express eq. (1) using index notation. [10%]

(c) In the presence of diffusion, the flux becomes $\mathbf{q} = \mathbf{a}\phi - \kappa \nabla \phi$, where κ is the scalar diffusion coefficient. Using conservation of ϕ for a domain V, show under what condition the governing partial differential equation for advective-diffusive transport is

$$\frac{\partial \phi}{\partial t} + \mathbf{a} \cdot \nabla \phi - \kappa \nabla^2 \phi = 0 \quad . \tag{2}$$

[30%]

(d) Derive a weak form for the steady-state version of eq. (2). Under what conditions will the solution of this steady-state version of eq. (2) correspond to a classical minimisation problem, and if the boundary condition is $\phi = 0$ on ∂V, what functional will it minimise? [30%]

4 A system of interest on the domain V is believed to be governed by the equation

$$-\nabla^2 u = f$$

with boundary condition $u = 0$ on the boundary ∂V. The observed response of the system is denoted by u_{obs}.

(a) To find f such that u approximates u_{obs} in a particular sense, stationary points of the functional

$$F = \int_V \left\{ \lambda\left(\nabla^2 u + f\right) + \frac{1}{2}\left(u - u_{\text{obs}}\right)^2 + \frac{\alpha}{2} f^2 \right\} dV, \tag{3}$$

where λ is a Lagrange multiplier and α is a constant, can be computed.

(i) Find the set of partial differential equations that must be solved to find the stationary points of F. [30%]

(ii) Subject to the constraint $\nabla^2 u + f = 0$, identify the expression J that this problem minimises. [10%]

(b) It is observed that the mean of u_{obs} over V is equal to β and we wish to ensure that the mean of u is equal to the mean of u_{obs}.

(i) Formulate a modified version of eq. (3) that satisfies this mean-value condition. [20%]

(ii) Find the set of equations, reduced to partial differential equations where possible, that must be solved to find the stationary points of the problem that satisfies the mean-value constraint. [40%]

END OF PAPER