Version PAD/2

EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 23 April 2019 9.30 to 11.10

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachments: 4A12 Data Card: (i) Vortex Dynamics (1 page);
(ii) Turbulence (2 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) A steady two-dimensional velocity field can be written as

$$\mathbf{u} = \left(\partial\psi/\partial y, -\partial\psi/\partial x \right)$$

where ψ is the streamfunction. Show that, if $\mathbf{n}$ is a unit normal to a streamline C, and $d\mathbf{r}$ part of that streamline, orientated in the direction of $\mathbf{u}$, then $\mathbf{u} \cdot d\mathbf{r} = \left| \nabla\psi \cdot \mathbf{n} \right| \left| d\mathbf{r} \right|$. [15%]

(b) The temperature field $T(\mathbf{x})$ in a steady two-dimensional flow is governed by

$$\mathbf{u} \cdot \nabla T = \alpha \nabla^2 T$$

where α is the thermal diffusivity, assumed constant. Show that, if α is small but finite, then $T = T(\psi)$ and hence

$$\nabla \cdot \left(T\mathbf{u} \right) = \nabla \cdot \left(\alpha \frac{dT}{d\psi} \nabla \psi \right)$$ [25%]

(c) Integrate this equation over the area bounded by a closed streamline C to show that, when α is small but finite,

$$\alpha \frac{dT}{d\psi} \oint_C \nabla \psi \cdot \mathbf{n} dr = 0$$

where the increment dr is part of the curve C and $\mathbf{n}$ is a unit normal to the streamline. Hence show that, for a steady flow with closed streamlines, T is independent of position in the limit of $\alpha \rightarrow 0$. [25%]

(d) State the Prandtl-Batchelor theorem, noting any restrictions that apply to it, and indicate how the analysis above may be adapted to prove this theorem. What is the physical interpretation of the theorem? [25%]

(e) In practice, the Prandtl-Batchelor theorem holds for statistically stationary turbulent flow whose mean streamlines are closed. Why is this? [10%]

2 (a) State Helmholtz's two laws of vortex dynamics. [20%]

(b) A magnetic field **B** in a highly conducting fluid is governed by

$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B}\cdot\nabla)\mathbf{u}, \quad \nabla\cdot\mathbf{B} = 0$$

Explain why Helmholtz's laws have their analogues in electromagnetism and state what you think these electromagnetic laws are. [20%]

(c) A short line element, $d\mathbf{r}$, which links two material points in a fluid is governed by

$$\frac{D}{Dt}d\mathbf{r} = (d\mathbf{r}\cdot\nabla)\mathbf{u}$$

Prove that magnetic field lines are frozen into a highly conducting fluid in the sense that they move with the fluid, like dye lines. [20%]

(d) The vector potential **A** for a magnetic field satisfies $\mathbf{B} = \nabla\times\mathbf{A}$, $\nabla\cdot\mathbf{A} = 0$. The magnetic helicity is defined in terms of **A** as $H = \int \mathbf{A}\cdot\mathbf{B}dV$, where the integral is over all space. Consider a magnetic field $\mathbf{B}(\mathbf{x}, t)$ consisting of two, thin, interlinked magnetic flux tubes. The fluxes of **B** in the two tubes are Φ_1 and Φ_2 and the centrelines of the tubes are C_1 and C_2. Confirm that the net magnetic helicity is given by

$$H = \Phi_1 \oint_{C_1} \mathbf{A}\cdot d\mathbf{r} + \Phi_2 \oint_{C_2} \mathbf{A}\cdot d\mathbf{r}$$

and use Stokes' theorem to show that, if the tubes are linked in a right-handed manner, then $H = 2\Phi_1\Phi_2$. [30%]

(e) It may be shown that the magnetic helicity for this configuration is an invariant. Use the results of Part (b) above to explain why this is the case. [10%]

3 Consider the idealisation of stationary homogeneous isotropic turbulence with integral lengthscale L and kinetic energy k moving with uni-directional mean velocity U.

(a) Discuss briefly the concept of the *energy cascade* and provide the rationale behind the estimate for the dissipation of energy ε given on the Data Card. [40%]

(b) The fluid is heated so that its kinematic viscosity changes from ν_1 to $\nu_1/2$ with L and k remaining the same. How do the dissipation, and the Taylor and the Kolmogorov lengthscales change? [30%]

(c) Define the integral timescale and discuss how you would measure it with a hot wire. What should the fastest frequency response of the hot wire be if it is to follow the fastest turbulent motions? Write down the relation between the integral timescale and the eddy turnover time and discuss its physical significance. [30%]

4 Consider a planar contraction in a rectangular duct. The width is not changing across the contraction while the height changes from H to $H/4$.

(a) By reference to the production terms in the Reynolds stress equation, suitably simplified for this flow, discuss how the three separate components of the turbulent intensity change across the contraction. [50%]

(b) Repeat the discussion in part (a), but now use conservation of angular momentum arguments. [50%]

END OF PAPER

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