Version RS/4

EGT3
ENGINEERING TRIPOS PART IIB

Monday 27 April 2015 2 to 3.30

Module 4F2

ROBUST AND NONLINEAR SYSTEMS AND CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
Engineering Data Book
CUED approved calculator allowed

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Define the structured singular value of a matrix M. [15%]

(b) A remotely piloted vehicle has two measured outputs and is to be controlled using two independent actuators. The "true" vehicle model is assumed to take the form

$$G_0(s)(I+\Delta(s)W_1(s)) \tag{1}$$

where G_0, Δ, W_1 are all 2×2 matrix transfer functions and Δ, W_1 are stable.

(i) Derive a necessary and sufficient condition for a controller $K(s)$ in the positive feedback convention to stabilise all plants of the form (1) for all Δ in H_∞ (unstructured) with $||\Delta||_\infty < 1$. State carefully, but do not prove, any results you use. [25%]

(ii) It is desired to design a controller $K(s)$ to achieve desired specifications for the transfer function $T_{d\to e}$ from d to e in Fig. 1 in which all signals are two-dimensional vectors. Draw an equivalent block diagram to Fig. 1 in which there is a 6×6 "generalised plant" P with inputs and outputs

$$\begin{bmatrix} w \\ d \\ u \end{bmatrix} \text{ and } \begin{bmatrix} z \\ e \\ y \end{bmatrix}$$

respectively, and there is an upper feedback block containing Δ and a lower feedback block containing K. Show the internal structure of P. [20%]

(iii) Show that the transfer function of P can be written in the form

$$\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & G_0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & I \end{bmatrix}$$

[20%]

(iv) Suppose the controller block K is combined with P to form a stable 4×4 transfer function M with inputs and ouputs

$$\begin{bmatrix} w \\ d \end{bmatrix} \text{ and } \begin{bmatrix} z \\ e \end{bmatrix}$$

and with the uncertainty represented as an upper feedback block containing Δ. State carefully, but do not prove, a necessary and sufficient condition to ensure $||T_{d\to e}||_\infty \le 1$ for all Δ in H_∞ (unstructured) with $||\Delta||_\infty < 1$. [20%]

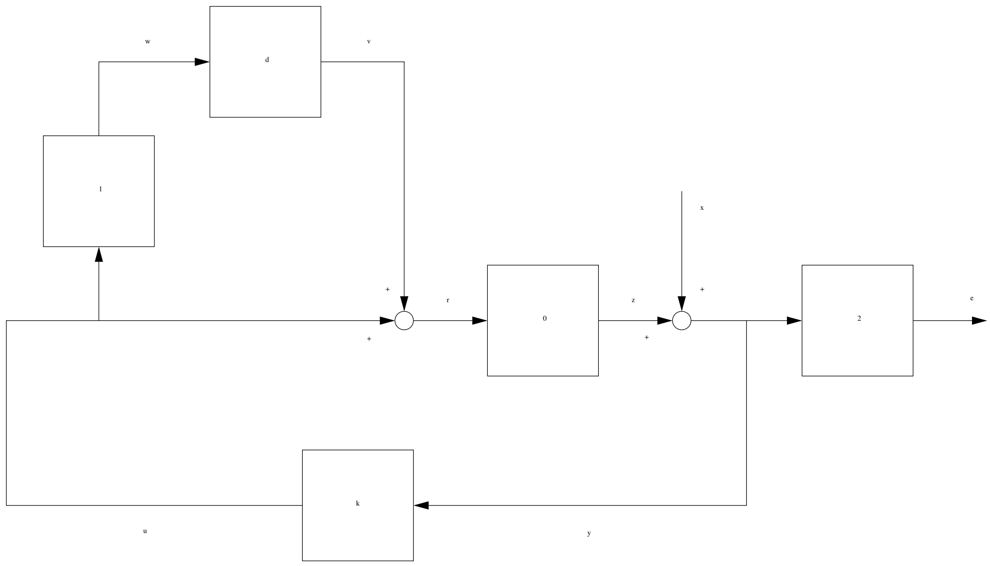

Fig. 1

2 (a) Let $G(s)$ and $K(s)$ be stable matrix transfer functions (not necessarily square) with the property that GK is a square matrix. Consider a closed-loop feedback system with positive feedback convention in which the plant and controller are modelled by G and K.

(i) State what is meant for the feedback system to be *internally stable*. [10%]

(ii) Show that the feedback system is internally stable if either $\|GK\|_\infty < 1$ or $\|KG\|_\infty < 1$. [20%]

(b) Let M be a complex matrix. State the relationship between the singular values of M and the eigenvalues of M^*M and MM^*. [10%]

(c) (i) Find the H_∞ norm of the transfer function

$$\frac{as}{(s+a)^2}$$

where a is a positive constant. [20%]

(ii) Let

$$G(s) = \begin{bmatrix} 1 \\ 10 \end{bmatrix} \frac{s}{s+a} \quad \text{and} \quad K(s) = \begin{bmatrix} 9 & -1 \end{bmatrix} \frac{a}{s+a}.$$

Find $\|G\|_\infty$, $\|K\|_\infty$, $\|GK\|_\infty$, and $\|KG\|_\infty$. With these choices of G and K, is the feedback system described in (a) internally stable? [30%]

(iii) What can be said about the internal stability of the feedback system if a negative feedback convention is used, or if a time delay is introduced into the loop at the plant input? [10%]

3 The dynamic model

$$m\ddot{x} + d\,\dot{x} + k\,(\sqrt{x^2+h^2} - l_0)x = 0$$

describes the motion of the mass-spring-damper system represented in Fig. 2 where m denotes the mass, d denotes the friction coefficient, k denotes the spring constant, l_0 denotes the spring natural length, and h denotes the distance between rod and spring attachment point.

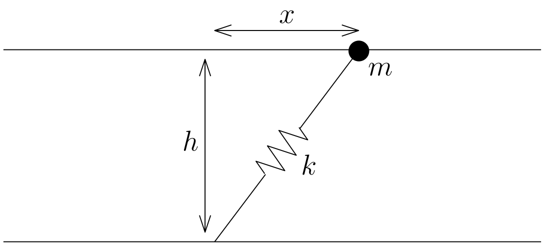

Fig. 2. A nonlinear mass-spring-damper system

(a) Determine the equilibria of the system as a function of the system parameters. Determine the physical property of the spring that will ensure a unique equilibrium configuration. [20%]

(b) Assuming a parameter configuration for which there is a unique equilibrium, prove by means of Lyapunov analysis that this equilibrium is globally asymptotically stable. State (but do not prove) and verify the assumptions of any theorem used to show the result. [40%]

(c) Assuming a parameter configuration for which there are multiple equilibria, analyze the (local) stability of each of them. [40%]

4 Goodwin (1965) proposed that the sequence of biochemical reactions represented in Fig. 3 can lead to limit cycle oscillations. In this figure, state variables x_i ($1 \leq i \leq n$) represent the concentration of different biochemical reactants while arrows represent biochemical reactions between them. We model each reaction $x_i \rightarrow x_{i+1}$ as a linear system with transfer function

$$H(s) = \frac{1}{\tau s + 1}$$

and the feedback loop from x_n to x_1 with the static nonlinearity

$$x_1 = \varphi(x_n) = \frac{2}{1 + x_n^k},\ k > 1.$$

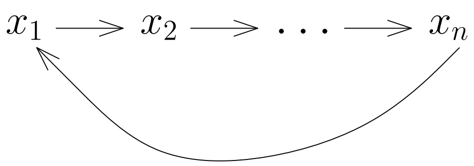

Fig. 3. Cyclic feedback biochemical pathway.

(a) Write a state-space model of the system in the positive orthant ($x_i \geq 0$ for all i) and determine its unique equilibrium $(\bar{x}_1, \dots, \bar{x}_n)$. [20%]

(b) Use the Nyquist stability criterion to study the stability of the equilibrium as a function of the parameters k, τ, and n. In particular, show that the equilibrium is always stable for $n = 2$, and that it is stable for $n = 3$ provided that $k < 16$. [40%]

(c) Rewrite the system in the shifted coordinates $z_i = x_i - \bar{x}_i$, $i = 1, \dots, n$. Considering the limiting case $k \rightarrow +\infty$, use the describing function method to predict the existence of a stable limit cycle and its amplitude whenever the equilibrium is unstable. [40%]

END OF PAPER