EGT0
ENGINEERING TRIPOS PART IA

Tuesday 9 June 2015 9 to 12

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

1 (**short**) Solve the difference equation

$$x_{n+2} - 5x_{n+1} + 4x_n = 0$$

with initial values of $x_0 = 1$ and $x_1 = 5$. [10]

2 (**short**)

(a) Using l'Hôpital's rule, find

$$\lim_{x\to 0}\left[\frac{\sin x - \sin^2 x}{x^2 - x + \sin^2 x}\right]$$

[5]

(b) Using the method of power series expansion, find

$$\lim_{x\to 0}\left[\frac{(\sin x - x)}{x^3} - \frac{1}{\cosh x}\right]$$

[5]

3 (**short**) For the matrix

$$A = \begin{bmatrix} 8 & 3 \\ 1 & 6 \end{bmatrix}$$

determine the eigenvalues and eigenvectors. [10]

4 (**long**)

(a) Find the general solution to the differential equation

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{I}{C}=0$$

describing the current response I of a resistor R, an inductor L, and a capacitor C connected in series for the following three cases:

(i) $R^2-4\frac{L}{C}>0;$

(ii) $R^2-4\frac{L}{C}=0;$

(iii) $R^2-4\frac{L}{C}<0.$ [10]

(b) Determine which case in (a) above gives an unstable solution for I. [5]

(c) Show that the current response I is oscillatory for $R=100\ \Omega$, $L=1\times10^{-2}$ H, and $C=2\times10^{-6}$ F, and determine the characteristic frequency. [8]

(d) If the initial conditions are $I=1$ A and $dI/dt=0$ at $t=0$, derive the complete solution for the current as a function of t. [7]

5 (**long**) A complex variable z satisfies

$$\frac{|z-i|}{|z+i|} = a$$

where a is a constant.

(a) Derive an equation for the locus of z on the Argand diagram. [12]

(b) Find the condition that a should satisfy such that z has valid solutions with positive values on the imaginary axis. [6]

(c) For $a = 0.9$, determine the intersections of the locus with the imaginary axis. [6]

(d) Sketch the locus of z on the Argand diagram for the value of a in part (c). [6]

SECTION B

6 (**short**) Using Laplace transforms, solve the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=1$$

with initial conditions $x=dx/dt=0$ at $t=0$. [10]

7 (**long**)

(a) The response $y(t)$ of a linear system to a general input $f(t)$ is given by the convolution integral

$$y(t)=\int_0^t f(\tau)\,g(t-\tau)\ d\tau$$

where $g(t)$ is the impulse response. Show that this integral and

$$y(t)=\int_0^t g(\tau)\,f(t-\tau)\ d\tau$$

are equivalent. [3]

(b) A linear system with input $f(t)$ is governed by the equation

$$\alpha\frac{dy}{dt}+y=f$$

where α is a constant.

(i) *Without using Laplace transforms,* find the step response of the system, assuming that $y(t)=0$ for $t<0$. [8]

(ii) Find the impulse response $g(t)$ of the system. [4]

(iii) Find the response to an input

$$f(t)=\begin{cases}\frac{1}{\beta}e^{-t/\beta} & t\geq 0\\ 0 & t<0\end{cases}$$

where $\beta<\alpha$. What happens to the response as $\beta\to 0$? Comment briefly on your result. [15]

8 (**short**) Assuming that the birth rate remains the same throughout the year and that people are born independently of each other, show that

$$P_n = \frac{d!}{(d-n)!\,d^n}$$

is the probability that n people chosen at random will all have different birthdays. The number of days in a year is d and n is less than d. [10]

9 (**short**) The function $f(x,y)$ is given by

$$f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$$

Find and classify the stationary points of $f(x,y)$. [10]

10 (**long**) An even function $f(t)$ is periodic with a period $T = 2$ and $f(t) = \cosh(t-1)$ for $0 \le t \le 1$.

(a) Sketch $f(t)$ in the range $-2 \le t \le 4$. [6]

(b) Show that

$$f(t) = \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty} \frac{\cos n\pi t}{(1+n^2\pi^2)}$$

is a Fourier series representation of $f(t)$. [15]

(c) Using the solution in (b), deduce that

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2} = \frac{1}{e^2-1}$$

[9]

SECTION C

11 (**short**) In computer graphics, a curved surface may be approximated by a mesh of small triangles. Consider the following C++ data structures for representing a mesh comprising 11119 triangles sharing 5580 vertices.

```
class vertex {
public:
  double x; // x coordinate of vertex
  double y; // y coordinate of vertex
  double z; // z coordinate of vertex
};
class triangle {
public:
  int v1; // v_list index for vertex 1
  int v2; // v_list index for vertex 2
  int v3; // v_list index for vertex 3
};
vertex v_list[5580]; // vertex list
triangle t_list[11119]; // triangle list
```

(a) If a `double` occupies 8 bytes and an `int` occupies 4 bytes, how much memory is required to store the mesh? [3]

(b) Write a short C++ code segment to calculate the centroid of the first triangle in the mesh. [7]

12 (**short**) The linear difference equation

$$f_i = f_{i-1} + f_{i-2}\ , \text{ with } f_0 = 0 \text{ and } f_1 = 1$$

generates the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8\ldots$ The equation has solution

$$f_i = \frac{(\lambda_1)^i - (\lambda_2)^i}{\sqrt{5}}, \text{ where } \lambda_1 = \frac{1+\sqrt{5}}{2} \text{ and } \lambda_2 = \frac{1-\sqrt{5}}{2}$$

Consider the following C++ methods for displaying the *n*th Fibonacci number.

```
// Method 1
int fibonacci(int n) {
  if (n<=1) return n;
  else return fibonacci(n-1) + fibonacci(n-2);
}
cout << fibonacci(n) << endl; // call to function in main program

// Method 2
float l_1 = (1.0 + sqrt(5.0))/2.0, l_2 = (1.0 - sqrt(5.0))/2.0;
cout << (pow(l_1, n) - pow(l_2, n))/sqrt(5.0) << endl;
```

For each method:

(a) State its algorithmic complexity. [5]

(b) Comment on its numerical accuracy. [5]

END OF PAPER

Part IA 2015

Paper 4: Mathematical Methods

Numerical Answers

Section A:

1 $x^n = \frac{4^{n+1}}{3} - \frac{1}{3}$

2 (a) -1 (b) -7/6

3 Eigen values are 9, 5; Eigen vectors are (3,1) and (-1, 1)

4 (c) f = 5000/2π Hz

5 (c) 9.526 ± 9.47

Section B:

9 (5, 0) – Minimum; (-5, 0) – Maximum; (3, 4) & (-3, -4) – Saddle points

Section C:

11 (a) 267348 bytes

12 (a) Method 1 is $O(2^n)$ (upper bound), Method 2 is $O(1)$