EGT0
ENGINEERING TRIPOS PART IA

Tuesday 7 June 2016 9 to 12

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

1 (**short**)

(a) Find the limit as $x \to 0$ of the expression $f(x) = (x\cos x - \sin x)/x^3$. [5]

(b) Find the limit as $x \to -4$ of the expression $g(x) = \sin(\pi x)/(x^2 - 16)$. [5]

2 (**short**)

(a) On an Argand diagram show all the solutions of the equation $z^6 = 64$. [5]

(b) On an Argand diagram sketch representative contours of $\mathrm{Im}(z^2) = \text{constant}$. [5]

3 (**short**)

(a) Show that the three vectors $\mathbf{a} = [3,-3,0]^{\mathrm{T}}$, $\mathbf{b} = [2,2,-1]^{\mathrm{T}}$, and $\mathbf{c} = [1,1,4]^{\mathrm{T}}$ are orthogonal, and hence find the volume of the box of which they form the sides. [5]

(b) Order the three vectors to make a right-handed set, i.e. bearing the same relationship to each other as the x, y and z axes in Cartesian coordinates. [5]

4 (**long**)

(a) Show that the coordinate transformation $\mathbf{U}$ mapping (x, y) into (x', y') such that

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

with $\theta = 45°$, maps the corners of a square defined by $(1,1)$, $(1,2)$, $(2,2)$, $(2,1)$ into the corners of another square with the same area. [10]

(b) (i) Show that $\mathbf{U}^4 = -\mathbf{I}$, where $\mathbf{I}$ is the identity matrix

(ii) Show that $\mathbf{U}^{-1} = \mathbf{U}^{\mathrm{T}}$. [5]

(c) If

$$\mathbf{A} = \begin{bmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{bmatrix}$$

and $a > 0$, find the three eigenvalues of $\mathbf{A}$ and the normalised eigenvector associated with the highest eigenvalue. What can one say about the direction of the other two eigenvectors? [15]

5 (**long**) A set of atoms $N(t)$ decay into "daughter" atoms at a rate given by $\lambda N(t)$ so that

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\lambda N(t)$$

where λ is a positive constant. We assume that $N = N_0$ at $t = 0$.

(a) Find an expression for $N(t)$, and show that after a time $T_{1/2} = \ln(2)/\lambda$ the number of the original atoms left has halved. [5]

(b) If the daughter atoms N_1 do not themselves decay, derive an expression for $N_1(t)$ assuming that $N_1(t) = 0$ at $t = 0$. [5]

(c) Suppose instead that the daughter atoms, N_1, also decay with a decay constant given by λ^* producing their own daughter atoms, N_2, which we assume do not decay. Show this means that

$$\frac{\mathrm{d}N_1(t)}{\mathrm{d}t} + \lambda^* N_1(t) = \lambda N_0 \exp(-\lambda t)$$

and, finding the particular integral for this equation for $N_1(t)$, solve for $N_1(t)$ under these new conditions, assuming again that $N_1(t) = 0$ at $t = 0$. [13]

(d) At $t = 0$, $N_1 = 0$, and as $t \to \infty$, $N_1(t) \to 0$. $N_1(t)$ is positive for $t > 0$. Derive an equation for the time, t^*, when the number of atoms $N_1(t^*)$ is a maximum. Show that t^* is positive where $\lambda > \lambda^*$ or $\lambda < \lambda^*$. [7]

SECTION B

6 (**short**) Using Laplace transforms, solve the differential equation

$$\frac{d^2x}{dt^2} - 4x = \sinh t$$

with initial conditions $x = dx/dt = 0$ at $t = 0$. [10]

7 (**long**) A linear system is described by

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = f(t)$$

with initial conditions $y = dy/dt = 0$ at $t = 0$.

(a) Show that the step response is

$$y(t) = \frac{1}{5} - e^{-t}\left[\frac{\cos 2t}{5} + \frac{\sin 2t}{10}\right]$$

and find the impulse response. [12]

(b) Using convolution, determine the system response to the input

$$f(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where α is a positive constant. What is the significance of the limit $\alpha \to 0$? Comment briefly on your result. [18]

8 (**short**)

(a) If the probability that a family has n children is $P_n = \alpha\, p^n$ with $n \geq 1$ then show that

$$P_0 = 1 - \frac{\alpha\, p}{1-p}$$

gives the probability that a family is childless. [5]

(b) Show that

$$\frac{1}{(1-x)^{n+1}} = \binom{n}{n} + \binom{n+1}{n} x + \binom{n+2}{n} x^2 + \cdots$$

using the binomial expansion. [5]

9 (**short**) The function $f(x,y)$ is given by

$$f(x,y) = 5x^3 - 2y^3 - 6x^2y + 54y.$$

Find and classify its stationary points. [10]

10 (**long**) An even function $f(t)$ is periodic with a period $T = 4$ and $f(t) = 4 - t^2$ for $0 \leq t \leq 2$.

(a) Sketch $f(t)$ in the range $-4 \leq t \leq 4$. [5]

(b) Show that

$$f(t) = \frac{8}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos\left(\frac{n\pi t}{2}\right)$$

is a Fourier series representation of $f(t)$. [15]

(c) Deduce that

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

using the expression in (b). [10]

SECTION C

11 (**short**) Consider the following C++ variables and data structures for representing the positions of pieces on a chessboard at any point.

```
enum PieceType {King, Queen, Bishop, Knight, Rook, Pawn, None};
enum PieceColour {Black, White};

class ChessPiece {
public:
  PieceType type;
  PieceColour colour;
};

ChessPiece board[8][8]; // 8 x 8 grid of locations
```

Write short segments of C++ code to:

(a) place a white Rook at a corner of the chessboard (any corner is acceptable); [2]

(b) print the locations (i.e. row and column) of all the Pawns on the board; [4]

(c) print the total number of black pieces on the board. [4]

12 (**short**) In the C++ code below, `f1` and `f2` are both designed to calculate a^n, where a is a real number and n is an integer. There is also a main program which calls `f1` and `f2`.

```
float f1(float a, int n) {
  float b = 1.0;
  while (n>0) {
    b = b*a;
    n = n-1;
  }
  return b;
}

float f2(float a, int n) {
  return exp(n*log(a));
}

int main() {
  cout << f2(7.0, 2) - f1(7.0, 2) << endl;
  return 0;
}
```

(a) What is the algorithmic complexity of `f1` in terms of n? What is the algorithmic complexity of `f2` in terms of n? [4]

(b) What, approximately, would you expect the main program to print, and why? [3]

(c) For what values of a and n will `f1` and `f2` fail catastrophically? [3]

END OF PAPER

Part IA 2016

Paper 4: Mathematical Methods

Numerical Answers

Section A:

1(a) -1/3; (b) $-\pi/8$

3 volume is 56

4(c) Eigenvalues are $-a$, $-a$, $2a$ Eigenvector for the largest eigenvalue is $\left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$

Section B:

9 (0, 3) – Maximum; (0, -3) – Minimum,

$(12/\sqrt{41}, 15/\sqrt{41})$ and $(-12/\sqrt{41}, -15/\sqrt{41})$ are saddle points

Section C:

12(b) the main program will display 7.62939e-06