Version HEMH/3

EGT2
ENGINEERING TRIPOS PART IIA

Monday 18 April 2016   2 to 3.30

Module 3C5

DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C5 Dynamics data sheet (6 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 A rigid body of mass $3m$ is made up of three equal square plates of side $2a$ welded together as shown in Fig. 1. Each plate has a vertex at the origin O. Plate P1 is normal to the x axis, plate P2 is normal to the y axis and plate P3 is normal to the z axis.

(a) Find the inertia matrix of the body at O referred to axes x, y, z and verify that the values of each of I_{xy}, I_{yz} and I_{xz} are equal in magnitude to ma^2. [40%]

(b) Show that (1,-1,0) and (1,1,-2) are both principal axes through O, find the other principal axis and verify that the body is "AAC". [30%]

(c) Find the coordinates of the centre of mass G and show that OG is a principal axis. Hence or otherwise find the principal moments of inertia at (0, $-a$, a). [30%]

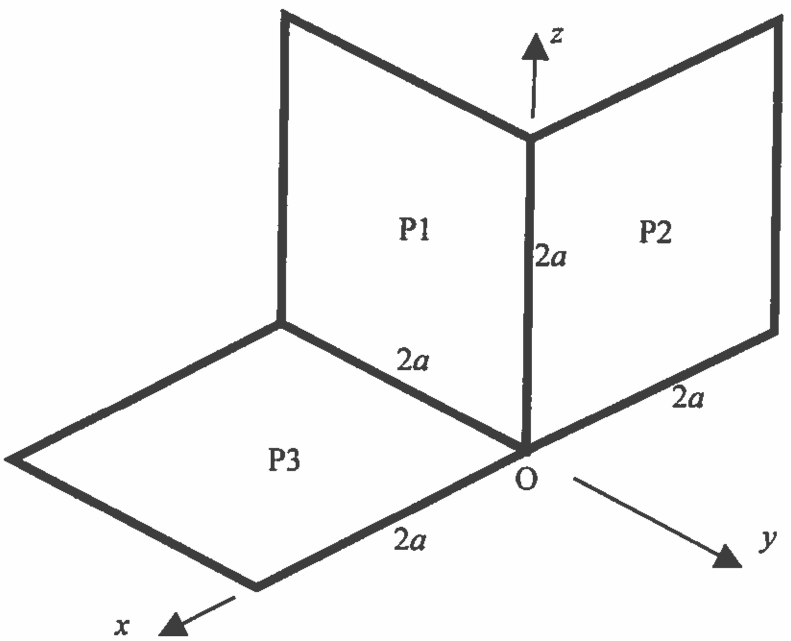


Fig. 1

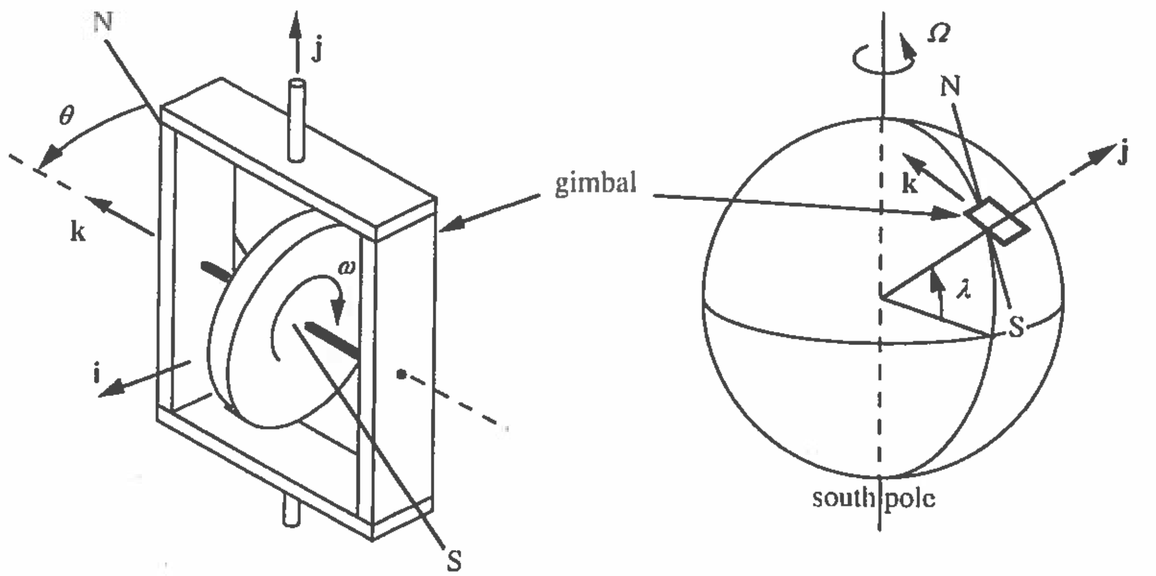
2 A simplified gyrocompass is shown in Fig. 2(a). The gimbal is free to turn about its vertical axis **j** through angle θ measured from North. The rotor has principal moments of inertia A, A, C at its centre and spins with 'fast' angular velocity ω about its axis **k**. Friction and gimbal inertia may be neglected. The Earth spins at constant angular velocity Ω and the instrument is situated at latitude λ as shown in Fig. 2(b).

(a) Determine the components of the Earth's spin in the gimbal-mounted axis system **i, j, k** and find expressions for the **i, j, k** components of angular velocity of the reference frame. [20%]

(b) Obtain an equation of motion for the gyrocompass. [40%]

(c) Show that there is a stable solution to this equation and explain how the gyrocompass can be used to point to True North. [20%]

(d) Find an expression for the frequency of small oscillations about True North. [20%]



(a)

(b)

Fig. 2

3 (a) A spinning top of mass m has principal moments of inertia A, A, C at its centre of mass which is at height a when the top is standing upright. It is spinning about its axis with angular speed ω.

(i) For the case of fast spin find an expression for the period of precession when the top is inclined at a small angle from the vertical. [20%]

(ii) Find the spin speed below which the top will not remain stably upright. [30%]

(b) The kinetic energy of a dynamic system consisting of N particles can be written

$$T = (1/2)\sum_{j=1}^{N} m_j \dot{q}_j^2 ,$$

where m_j is the mass of the jth particle and $\dot{q}_j$ is the velocity. The potential energy of the system is $V(q_1, q_2, \ldots, q_N)$ where q_j is the displacement of the jth particle. No external forces act on the particles.

(i) Derive an expression for the generalised momentum p_j of the jth particle. [10%]

(ii) By using Lagrange's equation derive the equations of motion of the system. [10%]

(iii) The Hamiltonian H is defined as the total energy of the system, and it is expressed in terms of p_j and q_j rather than $\dot{q}_j$ and q_j. Show that the equations derived in part (ii) are in agreement with Hamilton's equations, which state:

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} .$$

[15%]

(iv) From these equations show that the total energy of the system is constant. [15%]

4 A light circular hoop is mounted on a trolley and is able to spin about the vertical diameter, as shown in Fig. 3. The radius of the hoop is a, the rotation about the vertical axis is ψ and the horizontal translation of the trolley, which has mass M, is x. A bead of mass m is free to slide around the hoop, and the position of the bead is determined by the angle θ shown in Fig. 3. The trolley is attached to a linear spring of stiffness k. The acceleration due to gravity is g.

(a) Show that the kinetic energy of the bead is given by

$$T = (m/2)[a^2\dot{\theta}^2 + \dot{x}^2 cos^2\psi + 2a\dot{x}\dot{\theta}cos\theta cos\psi + (a\dot{\psi}sin\theta - \dot{x}sin\psi)^2] \quad .$$ [20%]

(b) Show that it is possible to define a generalised momentum for only one of the three degrees of freedom, and derive an expression for this momentum. [20%]

(c) If the hoop is forced to spin at a constant rate $\dot{\psi} = \Omega$ find the resulting equations of motion for x and θ and show that the system has two static equilibrium points. Show that only one of these can be stable, and stability is only obtained under the condition $g - a\Omega^2 > 0$. [30%]

(d) Find the linear equations of motion for small amplitude oscillations around the stable equilibrium point for the case $g - a\Omega^2 > 0$. Explain the various terms in the equations in physical terms. [30%]

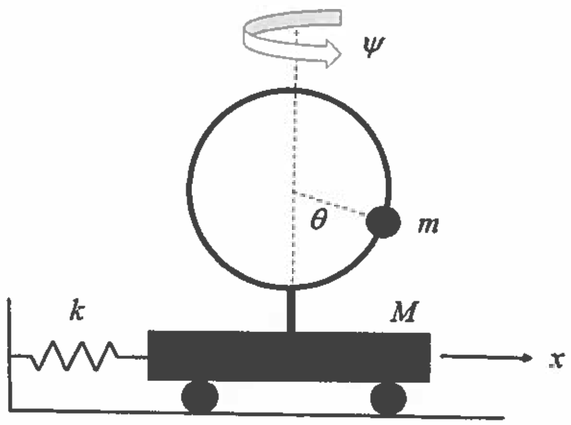


Fig. 3

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Part IIA Data sheet

Module 3C5 Dynamics

Module 3C6 Vibration

DYNAMICS IN THREE DIMENSIONS

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, M is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \mathbf{r})\,dm = \boldsymbol{I}\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and $A = \int (y^2 + z^2)\,dm$ $\quad B = \int (z^2 + x^2)\,dm$ $\quad C = \int (x^2 + y^2)\,dm$

$D = \int yz\,dm$ $\quad E = \int zx\,dm$ $\quad F = \int xy\,dm$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B - C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C - A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A - B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

VIBRATION MODES AND RESPONSE

Discrete systems

1. The forced vibration of an N-degree-of-freedom system with mass matrix M and stiffness matrix K (both symmetric and positive definite) is

$$M\,\ddot{\underline{y}} + K\,\underline{y} = \underline{f}$$

where $\underline{y}$ is the vector of generalised displacements and $\underline{f}$ is the vector of generalised forces.

2. Kinetic energy

$$T = \frac{1}{2}\dot{\underline{y}}^t M \dot{\underline{y}}$$

Potential energy

$$V = \frac{1}{2}\underline{y}^t K \underline{y}$$

3. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n{}^2 M\underline{u}^{(n)}\ .$$

4. Orthogonality and normalisation

$$\underline{u}^{(j)t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

5. General response

The general response of the system can be written as a sum of modal responses

$$\underline{y}(t) = \sum_{j=1}^{N} q_j(t)\,\underline{u}^{(j)} = U\underline{q}(t)$$

where U is a matrix whose N columns are the normalised eigenvectors $\underline{u}^{(j)}$ and q_j can be thought of as the "quantity" of the jth mode.

Continuous systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see p. 6 for examples.

$$T = \frac{1}{2}\int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See p. 4 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p. 4) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x)\,u_k(x)\,dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_j q_j(t)\,u_j(x)$$

where $w(x,t)$ is the displacement and q_j can be thought of as the "quantity" of the jth mode.

6. Modal coordinates $\underline{q}$ satisfy

$$\ddot{\underline{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\underline{q} = \underline{Q}$$

where $\underline{y} = U\underline{q}$ and the modal force vector

$$\underline{Q} = U^t \underline{f} \ .$$

Each modal amplitude $q_j(t)$ satisfies

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)\, u_j(x)\, dm$ and $f(x,t)$ is the external applied force distribution.

7. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x, and displacement w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

8. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

9. Impulse response

For a unit impulsive generalised force $f_j = \delta(t)$ the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

for $t \ge 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t \, e^{-\omega_n \zeta_n t}$$

for $t \ge 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin \omega_n t$$

for $t \ge 0$ (with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin \omega_n t \, e^{-\omega_n \zeta_n t}$$

for $t \ge 0$ (with small damping).

10. Step response

For a unit step generalised force $f_j = \begin{cases} 0 & t < 0 \\ 1 & t \ge 0 \end{cases}$ the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} [1 - \cos\omega_n t]$$

for $t \ge 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} [1 - \cos\omega_n t]$$

for $t \ge 0$ (with no damping), or

$$h(t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\frac{V}{\overline{T}} = \frac{\underline{y}^t K \underline{y}}{\underline{y}^t M \underline{y}}$ where $\underline{y}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p. 6.

If this quantity is evaluated with any vector $\underline{y}$, the result will be

(1) $\ge$ the smallest squared frequency;

(2) $\le$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{y}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\frac{V}{\overline{T}}$ is *stationary* near each mode.)

GOVERNING EQUATIONS FOR CONTINUOUS SYSTEMS

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.