Version VSD/3

EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 19 April 2016 9.30 to 11

Module 3C7

MECHANICS OF SOLIDS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C7 formulae sheet (2 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 A long cylindrical copper wire of radius b is heated by an electric current. The wire loses heat to the environment. When the temperature in the wire reaches a steady state, it is measured that the temperature T of the wire is only a function of the radial coordinate r:

$$T = (T_0 - T_\infty)(1 - r^2/b^2) + T_\infty,$$

where $T = T_0$ at the centre of the wire $r = 0$ and $T = T_\infty$ at the surface of the wire $r = b$. The temperature T_0 can be increased by increasing the electric current. For simplicity, assume that $T_\infty = 0$. The Young's modulus E, Poisson ratio ν, coefficient of thermal expansion α and tensile yield stress Y of copper are assumed to be independent of the temperature. You may assume plane strain conditions ($\varepsilon_{zz} = 0$).

(a) Neglecting any possible plasticity, determine the complete stress field in the copper wire. [25%]

(b) Assuming $\nu = 1/3$, sketch the stress components as functions of r/b. [25%]

(c) Calculate the radial displacement at the surface of the wire. [25%]

(d) Assuming that the material yields according to Tresca's criterion, determine the maximum T_0 that can be tolerated without causing plastic yielding anywhere in the wire. [25%]

2 A thin plate of height h and cross-sectional area A (Fig. 1) stands vertically on the ground. There is frictionless contact between the bottom of the plate and the ground, and the plate is free to move horizontally. The plate is made of an elastic material with density ρ, Young's modulus E, Poisson ratio ν and a coefficient of thermal expansion α. The plate is subjected to gravitational forces only, and hence is stress free at boundaries other than $y = 0$.

(a) Show that the stress field

$$\sigma_{xx} = 0, \quad \sigma_{yy} = \rho g y + c, \quad \sigma_{xy} = 0$$

satisfies the equilibrium equations, where g is the acceleration due to gravity. Hence determine the constant c from the boundary conditions. [25%]

(b) Again assuming plane stress conditions, determine the horizontal displacement u and vertical displacement v of the plate as functions of x and y. [25%]

(c) Calculate the total strain energy stored in the plate. [25%]

(d) The temperature of the plate is now increased uniformly by ΔT. Assuming that the stress field is still given by that determined in (a), obtain the new displacement field of the plate. [25%]

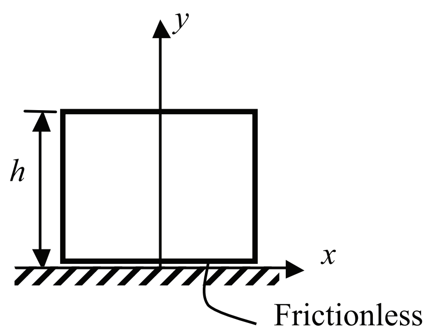

Fig. 1

3 (a) Explain why the upper bound theorem of plasticity is useful in modelling metal forming and shaping operations. [10%]

(b) Figure 2 shows a plane strain angular metal pressing process in which a work-piece is driven at speed v into an right-angled channel width a. The material of the work-piece is assumed to be rigid-perfectly plastic with a flow stress in shear of magnitude k. Tangential velocity discontinuities occur along the three thin lines indicated in the figure.

(i) Assume that the interfaces between the channel and the deforming material are well lubricated. Derive an expression for the upper bound driving pressure p in terms of h/a. [60%]

(ii) Hence determine minimum driving pressure p. [10%]

(iii) To consider the effect of friction between the channel and work-piece we assume that there exists an interfacial shear stress fk between the channel and the work-piece. Here f is the friction factor such that $0 < f < 1$. Derive an upper bound estimate for the driving pressure with effects of interfacial friction included. [20%]

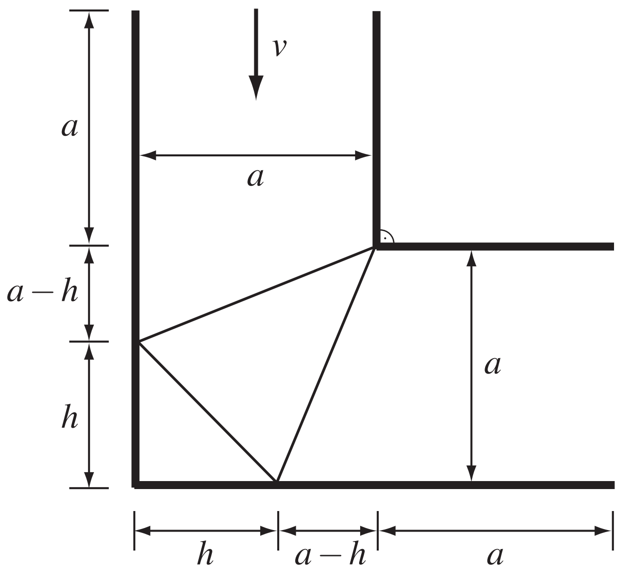

Fig. 2

4 Figure 3 shows an elastic cantilever OAB in the form of a triangular plate. The thickness t of the plate may be assumed to be much smaller than any other dimension. Side OA is horizontal, side AB is built into a rigid support and the angle AOB$= \alpha$. Side OA carries a uniformly distributed pressure of magnitude p. Employing polar coordinates with the origin at O and the angle θ measured downwards from OA, a suitable Airy stress function for this problem is

$$\phi = \frac{Cr^2}{\tan\alpha - \alpha}\left[\alpha - \theta + \frac{\sin 2\theta}{2} - \tan\alpha\cos^2\theta\right]$$

where C is a constant.

(a) Determine the stresses σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ at any point in the cantilever. [30%]

(b) State the boundary conditions along OA and OB and show that these boundary conditions are satisfied using the suggested stress function. Hence, determine the value of C in terms of the applied pressure p. [30%]

(c) If the angle α is small:

(i) show using simple beam theory, that at point A at the root of the cantilever, the stress $\sigma_{rr} = 3p/\alpha^2$. [20%]

(ii) show that the value of σ_{rr} at A using the above Airy stress function agrees with the simple beam theory prediction. [20%]

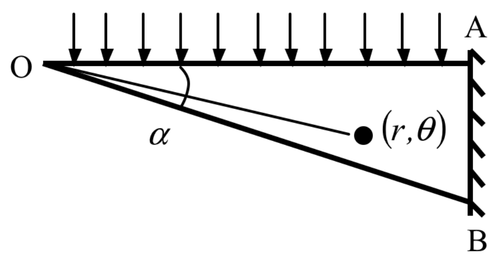

Fig. 3

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Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{\mathrm{d}(r\sigma_{rr})}{\mathrm{d}r} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{\mathrm{d}(r^2\sigma_{rr})}{\mathrm{d}r}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\displaystyle\int_c^r rT\mathrm{d}r$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\displaystyle\int_c^r rT\mathrm{d}r - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

Plane strain elastic constants $\quad \bar{E} = \dfrac{E}{1-\nu^2}\ ;\ \bar{\nu} = \dfrac{\nu}{1-\nu}\ ;\ \bar{\alpha} = \alpha(1+\nu)$

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y} = \dfrac{\partial^2\varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial\varepsilon_{rr}}{\partial r} + \dfrac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial\sigma_{yy}}{\partial y} + \dfrac{\partial\sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\} \times \left\{\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{\partial\psi}{\partial y}, \quad \sigma_{zy}\ (= \tau_y) = -\dfrac{\partial\psi}{\partial x}$

Equilibrium: $T = 2\int_A \psi\, dA$

Governing equation for elastic torsion: $\nabla^2\psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^T \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx}-\sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P{}^3 - I_1\,\sigma_P{}^2 + I_2\sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\left\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}}\left\{d\varepsilon_1{}^2 + d\varepsilon_2{}^2 + d\varepsilon_3{}^2\right\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1-\sigma_2|$, $|\sigma_2-\sigma_3|$ or $|\sigma_3-\sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1-d\varepsilon_2}{\sigma_1-\sigma_2} = \frac{d\varepsilon_2-d\varepsilon_3}{\sigma_2-\sigma_3} = \frac{d\varepsilon_3-d\varepsilon_1}{\sigma_3-\sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}.$$

Numerical answers

1. (d) $T_0 = \frac{2(1-\nu)Y}{E\alpha}$

2. (a) $c = -\rho g h$

   (c) $U = A\rho^{2g^2}h^3/(6E)$

3. (b)(ii) $p_{min} = 2k$

4. (b) $C = p/2$

Comments on questions

Q1 Axi-symmetric elastic fields

On the whole reasonably well attempted with the main errors being the students not converting the solution to plane strain and students making errors in using the Tresca yield criterion in determining the temperature at which first yield will occur.

Q2 Elastic fields in Cartesian co-ordinates

An unpopular question. The students that attempted it did generally very well with the main error being in integrating the fields to calculate the strain energy. Another mistake was in not applying the integration constants correctly while integrating the strain fields to determine displacements.

Q3 An upper bound plastic collapse calculation

While this question was popular the candidates typically made errors in drawing the velocity diagram. The students generally failed to recognise the continuity of velocity condition that was required to simply the velocity diagram and the associated algebra.

Q4 Airy stress function for a tapered cantilever beam

Very popular question that was extremely well attempted as seen by the high average mark. The most common error was related to messing up the sign conventions required to determine the constants in the Airy stress function.