EGT2
ENGINEERING TRIPOS PART IIA

Friday 1 May 2015 14.00 to 15.30

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3D7 Data Sheet (3 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Figure 1(a) shows the finite element model for a plate with a hole under axial tension. In attempting to solve this problem with a commercial finite element package the program stops with the error message 'the system matrix is singular'. Suggest the cause of this error message and how the finite element model can be fixed. [15%]

(b) For computing the shape functions of a three-noded triangular element we consider the function $T(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y$ with the parameters α_i to be determined. Explain why you should not alternatively use the function $T(x,y) = \alpha_0 + \alpha_1 x^2 + \alpha_2 y^2$. [15%]

(c) Consider the four-noded element shown in Fig. 1(b). The location of node 3 is (a,b).

(i) Show that the isoparametric map is given by

$$x = \frac{1}{4}(1+\xi)[2(1-\eta)+a(1+\eta)]$$
$$y = \frac{1}{4}(1+\eta)[2(1-\xi)+b(1+\xi)]$$

[25%]

(ii) Determine the conditions on the location of node 3 by considering the Jacobian at the point with the coordinates $\xi = 1$ and $\eta = 1$ in the parent element. [25%]

(d) Determine the shape functions N_1 and N_5 of the five-node element shown in Fig. 1(c). [20%]

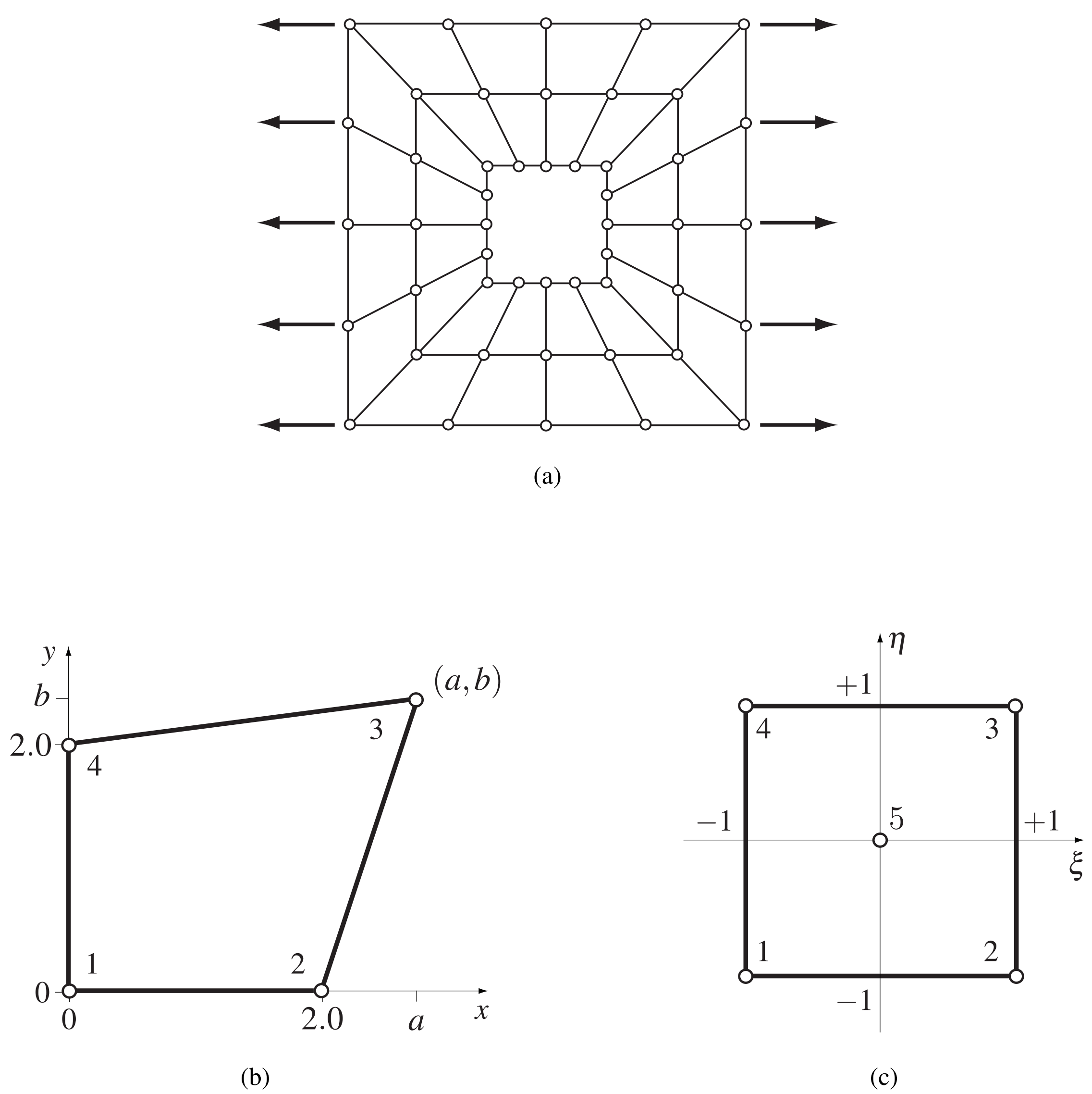


Fig. 1

2 On the domain Ω with boundary Γ, the following modified two-dimensional heat equation is given

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+c\frac{\partial T}{\partial x}+s=0\,,$$

where T is the temperature, k and c are constants and s is a prescribed domain heat source. The temperature boundary condition over the entire boundary is $T=0$.

(a) Derive the weak form of this modified heat equation. [30%]

(b) The following integral is to be discretised and evaluated

$$c\int_\Omega w\frac{\partial T}{\partial x}\,d\Omega\,,$$

where c is a constant and w is the test function.

(i) For the triangular three-node element shown in Fig. 2(a), compute the (1,2) and (2,1) components of the resulting matrix. [45%]

(ii) For the four-node quadrilateral element shown in Fig. 2(b) what is the minimum number of Gauss quadrature points needed in the ξ and η directions to perform exact integration. Justify your answer. [25%]

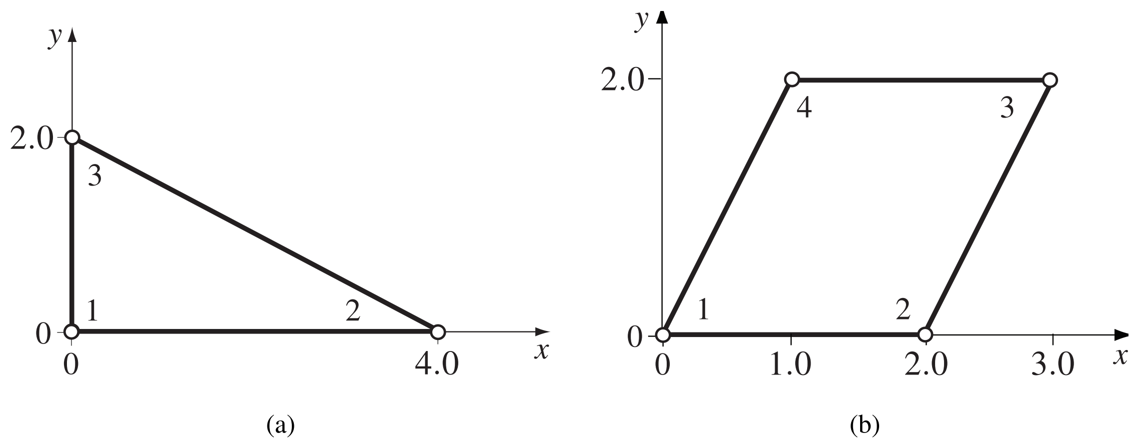


Fig. 2

3 A particular problem on the domain $(0, L)$ involves finding the function u such that

$$\int_0^L \alpha \frac{d^2 v}{dx^2} \frac{d^2 u}{dx^2}\, dx + \int_0^L \beta v u\, dx + \gamma v(u - g)|_{x=L} = 0$$

for all admissible test functions v, where α, β, γ and g are prescribed constants.

(a) Identify the strong form of the differential equation that is satisfied by a solution of this problem and identify any boundary conditions that are implied. [40%]

(b) The appropriate shape functions for this problem are Hermitian shape functions. Explain briefly why Hermitian shape functions are required for this problem. [10%]

(c) Sketch the Hermitian shape functions for a two-node element. [20%]

(d) Formulate a system of equations for computing the Hermitian shape functions for an element that runs from x_1 to x_2 ($x_1 < x_2$). [20%]

(e) If this problem is solved using two-node Hermitian elements and Gauss quadrature, how many quadrature points would you recommend using? Provide a brief justification for your answer. [10%]

4 A particular time dependent problem involves the semi-discrete finite element system

$$\boldsymbol{M}\ddot{\boldsymbol{a}} + \alpha \boldsymbol{K}\dot{\boldsymbol{a}} + \boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$$

where $\boldsymbol{M}$ is the mass matrix, $\boldsymbol{K}$ is the stiffness matrix, $\alpha \geq 0$ is a constant and $\boldsymbol{f}$ is prescribed.

(a) Using the Verlet time stepping scheme

$$\begin{aligned} y_{n+1} &= y_n + \Delta t \dot{y}_n + \frac{\Delta t^2}{2} \ddot{y}_n \\ \dot{y}_{n+1} &= \dot{y}_n + \frac{\Delta t}{2} (\ddot{y}_{n+1} + \ddot{y}_n) \end{aligned}$$

formulate a fully discrete version of the finite element problem. [50%]

(b) Would you recommend this time stepping scheme for $\alpha > 0$? Briefly justify your answer. [15%]

(c) Why is a lumped mass matrix appropriate for this problem, and how can a lumped mass matrix be constructed? [15%]

(d) Consider a scenario in which this problem is solved using an implicit time integration scheme and the linear solver has complexity $O(n^2)$, where n is the number of degrees of freedom. Estimate the change in the solution time per time step for 2D and 3D problems if the mesh is refined such that the number of elements in each direction is doubled. [20%]

END OF PAPER

3D7 DATA SHEET

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T \boldsymbol{f}\, dV$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Heat flux	$\boldsymbol{q} = -\boldsymbol{D}\nabla T$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI \boldsymbol{B}\, dV$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

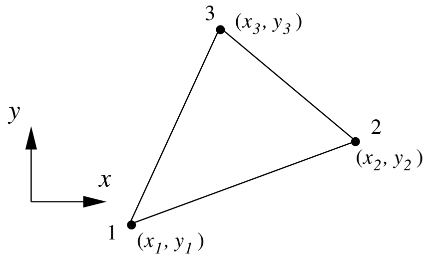
2D plane stress

$$\boldsymbol{D} = \frac{E}{\left(1-\nu^2\right)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D, isotropic)

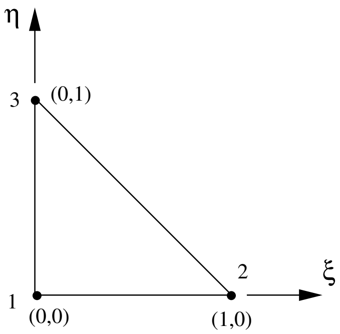
$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Shape functions

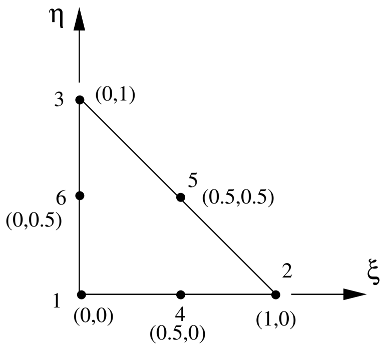


$$N_1 = \left((x_2 y_3 - x_3 y_2) + (y_2 - y_3)\,x + (x_3 - x_2)\,y\right)/2A$$
$$N_2 = \left((x_3 y_1 - x_1 y_3) + (y_3 - y_1)\,x + (x_1 - x_3)\,y\right)/2A$$
$$N_3 = \left((x_1 y_2 - x_2 y_1) + (y_1 - y_2)\,x + (x_2 - x_1)\,y\right)/2A$$

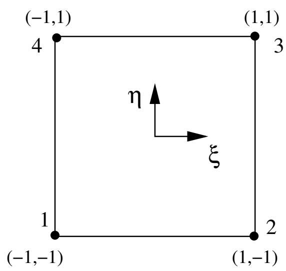
$A =$ area of triangle



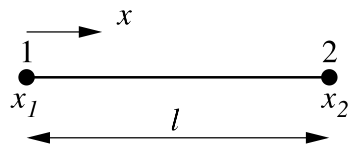
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2\,(1 - \xi - \eta)^2 - (1 - \xi - \eta)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi\,(1 - \xi - \eta)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta\,(1 - \xi - \eta)$$



$$N_1 = (1 - \xi)\,(1 - \eta)/4$$
$$N_2 = (1 + \xi)\,(1 - \eta)/4$$
$$N_3 = (1 + \xi)\,(1 + \eta)/4$$
$$N_4 = (1 - \xi)\,(1 + \eta)/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2(-l+2(x_1-x))}{l^3}$$

$$M_1 = \frac{(x-x_1)(x-x_2)^2}{l^2}$$

$$N_2 = \frac{(x-x_1)^2(l+2(x_2-x))}{l^3}$$

$$M_2 = \frac{(x-x_1)^2(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1,1)$

Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$

Numerical Answers

1. Condition on location of node 3: $a+b>2$

2. $K_{12}=1/3$, $K_{21}=-1/3$

4. Change in solution time: $16\,(2D)$, $64\,(3D)$