EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 27 April 2016 2 to 3:30

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3D7 Data Sheet (3 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Figure 1(a) shows a finite element mesh consisting of one three-noded triangular element. The element represents a planar elastic sheet with Young's modulus $E = 200$ and Poisson's ratio $\nu = 0.3$ under plane stress conditions. In the following consider only the unconstrained degrees of freedom.

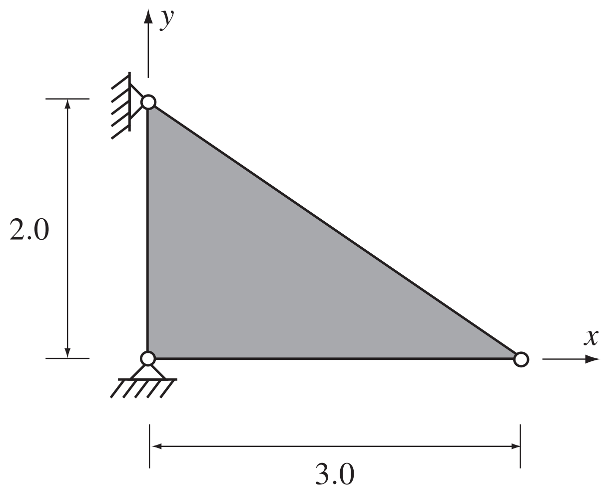
(i) Determine the strain-displacement matrix $\boldsymbol{B}^e$. [10%]

(ii) Determine the stiffness matrix. [30%]

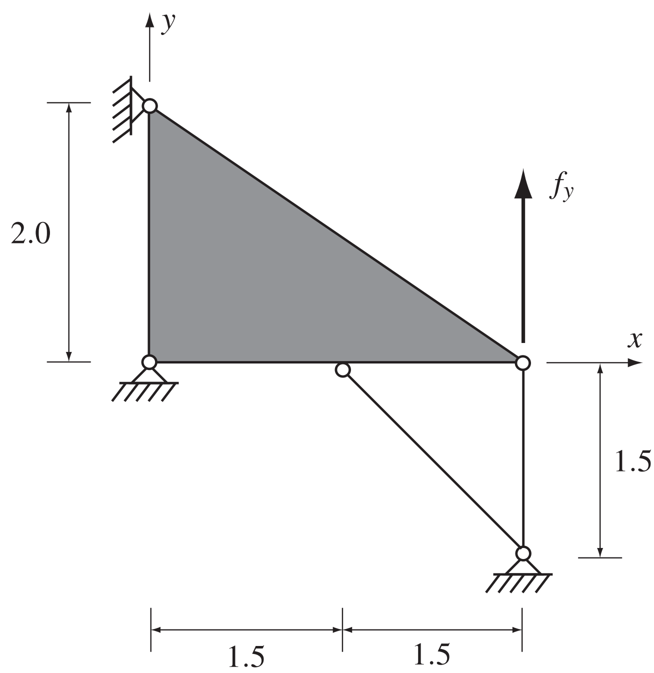
(b) Figure 1(b) shows a three-noded triangular element loaded with a force f_y and supported by two two-noded one-dimensional bar elements. The element's material parameters are as in part (a). The product of the Young's modulus and cross-sectional area is $EA = 100$ for the two bars.

(i) Determine the stiffness matrix of the two bar elements. Consider only the unconstrained degrees of freedom. [20%]

(ii) Determine the displacement of the node at which the load is applied. [40%]



(a)



(b)

Fig. 1

2 (a) Figure 2(a) shows the finite element model for a simply supported plate with a concentrated force. Solving the problem gives the nonphysical displacements shown in Fig. 2(b). Suggest the cause of this error and how it can be fixed. [10%]

(b) The finite element approximation u^h to a prescribed function f is determined by solving

$$\int_\Omega (u^h - f) w^h \, d\Omega = 0$$

with

$$u^h = \sum_{I=1}^{n_{np}} N_I u_I \quad \text{and} \quad w^h = \sum_{I=1}^{n_{np}} N_I w_I \,,$$

where w^h is the weight function, n_{np} is the number of nodes in the mesh, N_I are the global shape functions, and u_I and w_I are the nodal values.

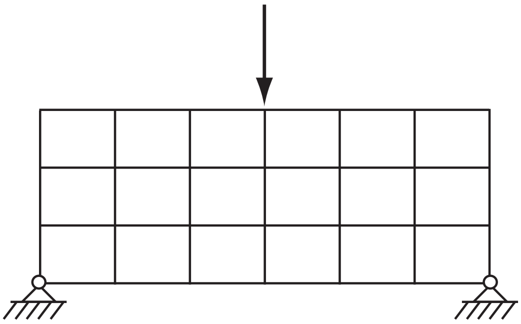
(i) Derive the discrete system of equations for computing the nodal values u_I. Clearly identify the resulting matrices and vectors. [35%]

(ii) The discrete system of equations obtained in (i) can also be derived from the least squares fit of u^h to f. Give the expression for the least squares error, and from this derive the discrete system of equations for computing the nodal values u_I. [25%]

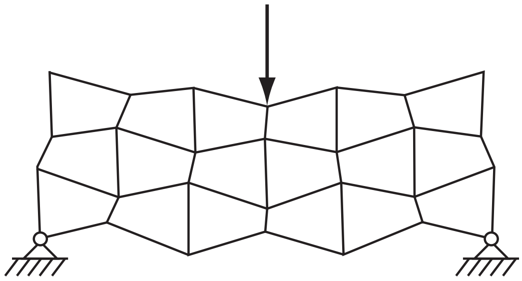
(c) Consider on a two-dimensional domain the weak form

$$\int_\Omega (\nabla u)^T \nabla w \, d\Omega + \int_\Omega u \boldsymbol{c}^T \nabla w \, d\Omega + \int_\Omega (u - f) w \, d\Omega = 0 \,,$$

where u is the unknown variable, w is the weight function, $\boldsymbol{c}$ is a prescribed vector and f is a prescribed function. Determine the corresponding strong form without the boundary terms. [30%]



(a)



(b)

Fig. 2

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3 For an elastic rod that runs from $x = 0$ to $x = L$, the equilibrium equation is

$$-\frac{d}{dx}\left(E(x)\frac{du}{dx}\right) = f\,,$$

where $E(x) > 0$ is the Young's modulus, u is the displacement and f is a constant distributed force. At $x = 0$, the boundary condition $u = 0$ is applied.

(a) For the case of a Neumann boundary condition $E(x)(du/dx) = h$ at $x = L$, where h is given:

(i) Derive the weak formulation of this problem. [20%]

(ii) For a linear element of length l and constant E, formulate the element stiffness matrix and element RHS vector. [20%]

(iii) If E varies linearly with x and Gauss quadrature is used to compute the element matrix and vector, for a linear element how many quadrature points would you recommend? [10%]

(b) For the case of a Robin boundary condition $au + E(x)(du/dx) = g$ at $x = L$, where a and g are given constants.

(i) Derive the weak formulation of this problem. [20%]

(ii) Formulate the element stiffness matrix and element RHS vector for the element that includes the boundary at $x = L$. The length of the element is l. Assume constant E. [20%]

(iii) Provide a physical interpretation of the parameter a. [10%]

4 The θ-method applied to the model problem

$$\frac{dy}{dt} = -\lambda y,$$

where $\lambda > 0$, is given by

$$\frac{y_{n+1} - y_n}{\Delta t} = -\lambda((1-\theta)y_n + \theta y_{n+1}),$$

where Δt is the time step.

(a) Show that the θ-method is unconditionally stable when $\theta \geq 1/2$. [40%]

(b) For unsteady heat conduction, the semi-discrete finite element problem has the form

$$\boldsymbol{M}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}.$$

(i) Use the θ-method to devise a scheme to compute $\boldsymbol{u}_{n+1}$. [20%]

(ii) When $\theta = 0$, the critical time step for the model problem is $2/\lambda$. If the matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ have the form:

$$\boldsymbol{K} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{M} = \begin{bmatrix} 6 & 1 \\ 1 & 5 \end{bmatrix}$$

Use model analysis to determine the critical time step for this problem when $\theta = 0$. [30%]

(iii) Comment on the suitability of the θ-method for heat conduction problem when $\theta < 1/2$. [10%]

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3D7 DATA SHEET

Element relationships

Elasticity

Displacement	$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}_e$
Strain	$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{a}_e$
Stress (2D/3D)	$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$
Element force vector (body force only)	$\boldsymbol{f}_e = \int_{V_e} \boldsymbol{N}^T \boldsymbol{f}\, dV$

Heat conduction

Temperature	$T = \boldsymbol{N}\boldsymbol{a}_e$
Temperature gradient	$\nabla T = \boldsymbol{B}\boldsymbol{a}_e$
Heat flux	$\boldsymbol{q} = -\boldsymbol{D}\nabla T$
Element conductance matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B}\, dV$

Beam bending

Displacement	$v = \boldsymbol{N}\boldsymbol{a}_e$
Curvature	$\kappa = \boldsymbol{B}\boldsymbol{a}_e$
Element stiffness matrix	$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}^T EI\boldsymbol{B}\, dV$

Elasticity matrices

2D plane strain

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

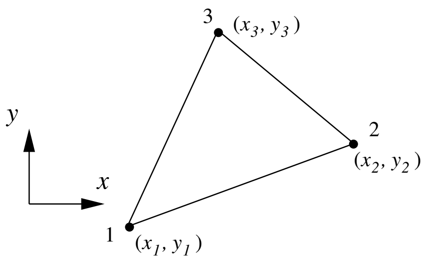
2D plane stress

$$\boldsymbol{D} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

Heat conductivity matrix (2D, isotropic)

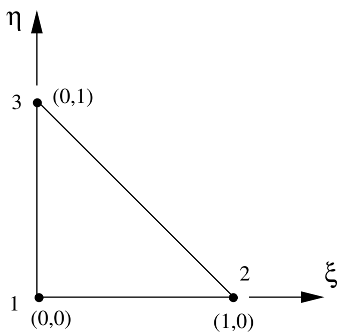
$$\boldsymbol{D} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Shape functions

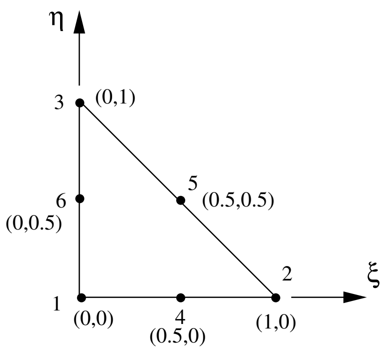


$$N_1 = ((x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y)/2A$$
$$N_2 = ((x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y)/2A$$
$$N_3 = ((x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y)/2A$$

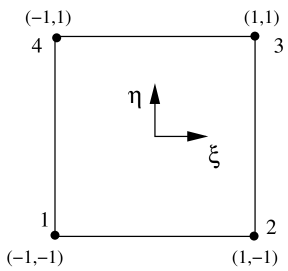
$A =$ area of triangle



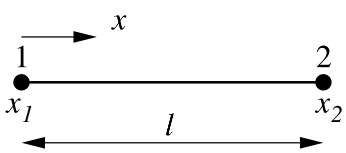
$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



$$N_1 = 2(1 - \xi - \eta)^2 - (1 - \xi - \eta)$$
$$N_2 = 2\xi^2 - \xi$$
$$N_3 = 2\eta^2 - \eta$$
$$N_4 = 4\xi(1 - \xi - \eta)$$
$$N_5 = 4\eta\xi$$
$$N_6 = 4\eta(1 - \xi - \eta)$$



$$N_1 = (1 - \xi)(1 - \eta)/4$$
$$N_2 = (1 + \xi)(1 - \eta)/4$$
$$N_3 = (1 + \xi)(1 + \eta)/4$$
$$N_4 = (1 - \xi)(1 + \eta)/4$$



Hermitian element

$$N_1 = \frac{-(x-x_2)^2(-l+2(x_1-x))}{l^3}$$

$$M_1 = \frac{(x-x_1)(x-x_2)^2}{l^2}$$

$$N_2 = \frac{(x-x_1)^2(l+2(x_2-x))}{l^3}$$

$$M_2 = \frac{(x-x_1)^2(x-x_2)}{l^2}$$

Gauss integration in one dimension on the domain $(-1,1)$

Using n Gauss integration points, a polynomial of degree $2n-1$ is integrated exactly.

number of points n	location ξ_i	weight w_i
1	0	2
2	$-\frac{1}{\sqrt{3}}$	1
	$\frac{1}{\sqrt{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$\sqrt{\frac{3}{5}}$	$\frac{5}{9}$