EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 26 April 2017 2 to 3.30

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3D7 Data Sheet (3 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 For an elastic rod that runs from $x = 0$ to $x = L$, the equilibrium equation for a particular problem is:

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f$$

where $EA > 0$ is constant, u is the displacement and $f = -\alpha u$ is a distributed force, with $\alpha \geq 0$ being constant.

(a) For the case $u = 0$ at $x = 0$ and $EA(du/dx) = F$ at $x = L$:

(i) Derive the weak formulation of this problem. [20%]

(ii) For the cases of linear and quadratic elements, what quadrature scheme would you recommend, and how many points would you use? [20%]

(iii) Describe how the Dirichlet boundary condition could be enforced. [10%]

(b) The elastic rod is now broken into two at $x = L/3$, and the two sections are connected by a spring with constant stiffness $k \geq 0$. The spring is considered to be of negligible length.

(i) Derive the weak formulation of this problem. [40%]

(ii) Under what conditions would you expect the stiffness matrix for this problem to be singular? [10%]

2 (a) Following an elastic finite element computation on a structured mesh with uniform element size h, the mesh is refined uniformly such that the error in the L^2 (displacement) norm is reduced by a factor of two.

(i) For both linear and quadratic elements, estimate the factor by which the cell size measure h needs to be reduced in order to halve the error in the displacement norm. Consider two- and three-dimensional cases. [20%]

(ii) The linear system $\boldsymbol{Ku} = \boldsymbol{b}$ is solved using a linear solver with time cost $O(n^2)$, where n is the number of degrees of freedom for the system. In three dimensions and using linear elements, estimate the increase in computational time required to halve the displacement error. [20%]

(iii) Comment qualitatively on the necessary conditions for the standard *a priori* error estimates to hold. [10%]

(b) For an unsteady heat conduction problem, the semi-discrete problem obtained from a structured mesh with uniform element size has the form:

$$\boldsymbol{M}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{b} \tag{1}$$

where $\boldsymbol{M}$ is the mass matrix and $\boldsymbol{K}$ is the 'stiffness matrix'.

(i) If a mesh is refined such that the number of degrees-of-freedom, n, is doubled, estimate the increase in time for assembling $\boldsymbol{M}$ and $\boldsymbol{K}$. [10%]

(ii) An explicit time stepping scheme is used to solve Eqn. (1) for a large, three-dimensional problem using a time step that is just below the critical time step. The mesh is then refined uniformly such that the number of cells increases by a factor of 8. Provide a lower bound estimate for the increase in the computation time for the refined problem relative to the coarser problem. [30%]

(iii) Would you recommend an explicit or implicit method for this problem? Explain your answer, and summarise the key differences between explicit and implicit methods. [10%]

3 Figure 1 shows a three-node isoparametric element K and the corresponding parent element $\hat{K}$. The coordinates of the three nodes of K are $(x_1, y_1) = (2, 1)$, $(x_2, y_2) = (5, 3)$ and $(x_3, y_3) = (3, 4)$. A point P in the isoparametric element corresponds to the quadrature point $(1/3, 1/3)$ of the one-point integration over $\hat{K}$.

(a) Write down the linear transformation $\boldsymbol{F}$ from $\hat{K}$ to K, hence compute the coordinates of the point P in the isoparametric element. [10%]

(b) Evaluate the Jacobian matrix $\boldsymbol{J}$,

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

at point P. [20%]

(c) The shape functions N_i of the six-node triangular element on K are defined via the unique correspondence through the linear transformation $\boldsymbol{F}$ as $N_i(x, y) = N_i(\xi, \eta)$, as shown in Fig. 2. Express $(\partial N_i/\partial x, \partial N_i/\partial y)$ in terms of $(\partial N_i/\partial \xi, \partial N_i/\partial \eta)$. [20%]

(d) Calculate the shape function N_5 on K of node 5 of the six-node element. [20%]

(e) The integration weight of the one-point integration over the parent element is $\frac{1}{2}$. Use the one-point integration to compute

$$\int_K \frac{\partial N_4}{\partial x} \frac{\partial N_4}{\partial x} dxdy \, .$$

[30%]

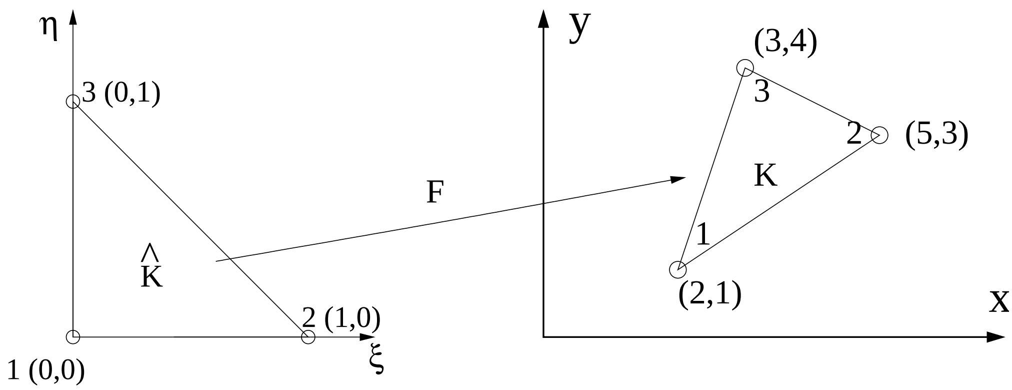

Fig. 1

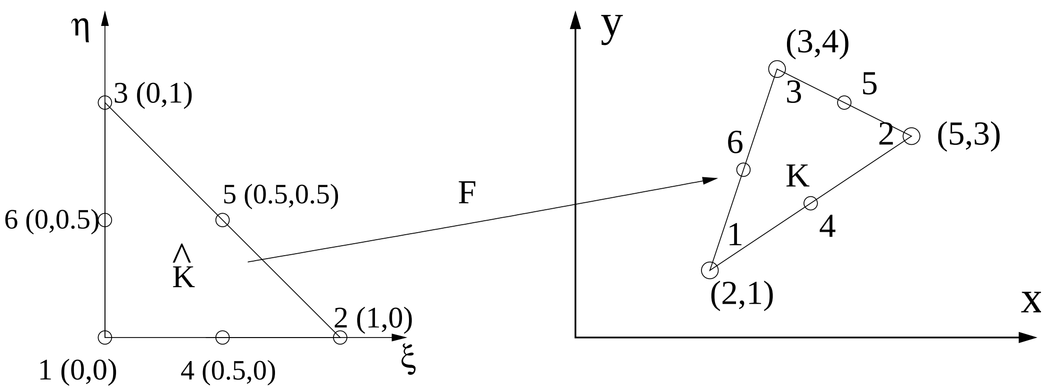

Fig. 2

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4 Consider the heat equation

$$\nabla^2 u = -1 \tag{2}$$

in the square whose vertices are at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. The boundary conditions are

$$u(x,0) = u(0,y) = u(1,y) = 0, \quad u(x,1) = x - x^2 \ .$$

The square is divided into four square elements as shown in Fig. 3. We seek the finite element solution of Eqn. (2) using the four-node rectangular elements associated with these four squares. When answering the following questions, make use of symmetry when applicable.

(a) Write down the weak formulation of Eqn. (2). [10%]

(b) Let $\boldsymbol{N}^e$ be the shape functions of the four-node rectangular element [1] in Fig. 3. For this element:

(i) Calculate the $\boldsymbol{B}^e$ matrix. [20%]

(ii) Calculate the element conductance matrix

$$\boldsymbol{K}^e = \int_{[1]} \boldsymbol{B}^{eT} \boldsymbol{B}^e d\Omega \ .$$

[20%]

(iii) Calculate the element source flux vector

$$\boldsymbol{f}^e = \int_{[1]} \boldsymbol{N}^{eT} d\Omega \ .$$

[20%]

(c) Find the finite element solution of Eqn. (2) using four-node rectangular elements. [30%]

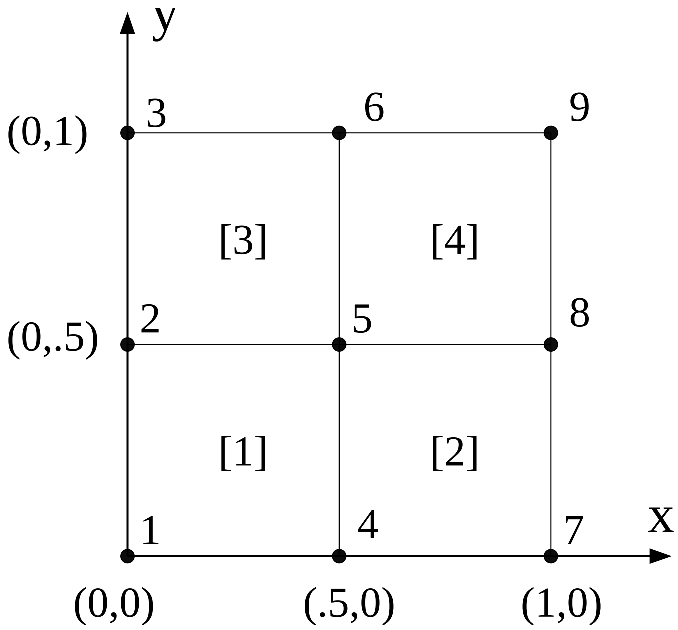

Fig. 3

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