Version JS/3

EGT2
ENGINEERING TRIPOS PART IIA

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Tuesday 28 April 2015 9.30 to 11

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**Module 3F1**

**SIGNALS AND SYSTEMS**

*Answer not more than* ***three*** *questions.*

*The* ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

*Answers to questions in each section should be tied together and handed in separately.*

*Write your candidate number* ***not*** *your name on the cover sheet.*

**STATIONERY REQUIREMENTS**
Single-sided script paper

**SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM**
CUED approved calculator allowed
Engineering Data Book

## 10 minutes reading time is allowed for this paper.

## You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Consider the discrete-time system given by the following difference equation

$$y_{k+1} = \frac{1}{2}y_k + u_k + d_k, \quad y_0 = 0 \tag{1}$$

where $y_k$ is the output, $u_k$ is the input and $d_k$ is a disturbance that we wish to attenuate. Consider also the feedback controller given by

$$u_{k+1} = au_k - \frac{1}{4}y_k, \quad u_0 = 0 \tag{2}$$

where $a$ is a constant to be determined.

(a) From equation (1), find the transfer functions $G(z)$ and $P(z)$ in $Y(z) = G(z)U(z) + P(z)D(z)$. [20%]

(b) From equation (2), find the transfer function of the controller $K(z)$ from $Y(z)$ to $U(z)$. [10%]

(c) Show that the closed-loop transfer function $M(z)$ from $D(z)$ to $Y(z)$ is given by

$$M(z) = \frac{z-a}{(z-\frac{1}{2})(z-a)+\frac{1}{4}}$$

[20%]

(d) Assume that the disturbance $d_k$ is a step. Find the value of the constant $a$ that minimises the effect of the disturbance of the closed-loop system in steady-state. [30%]

(e) Let $a = 0$ and assume now that the disturbance is given by $d_k = \cos(k)$. Determine the behaviour of $y_k$ as $k$ becomes large. [20%]

2 (a) (i) Consider the four frequency plots in Fig. 1. Indicate which one corresponds to the transfer function

$$G(z) = \frac{0.5}{(z-0.5)z}$$

Justify your answer. [30%]

(ii) Sketch the frequency plot (magnitude and phase) of $G(z)^{10}$. [20%]

(b) A random variable $Y$ is to be generated from a random variable $X$ using the monotonic increasing function $Y = g(X)$.

(i) Show that $f_Y(y)$, the probability density function (pdf) of $Y$, is related to $f_X(x)$, the pdf of $X$, by

$$f_Y(y) = \frac{f_X(x)}{g'(x)}$$

where $y = g(x)$ and $\frac{dy}{dx} = g'(x)$. [25%]

(ii) Let $X$ be uniformly distributed over $(-\frac{\pi}{2}, \frac{\pi}{2})$. Determine $g(x)$ such that $Y$ has a Cauchy pdf

$$f_Y(y) = \frac{1}{\pi(1+y^2)}$$

[Hint: $\frac{d\tan x}{dx} = 1 + \tan^2 x$.] [25%]

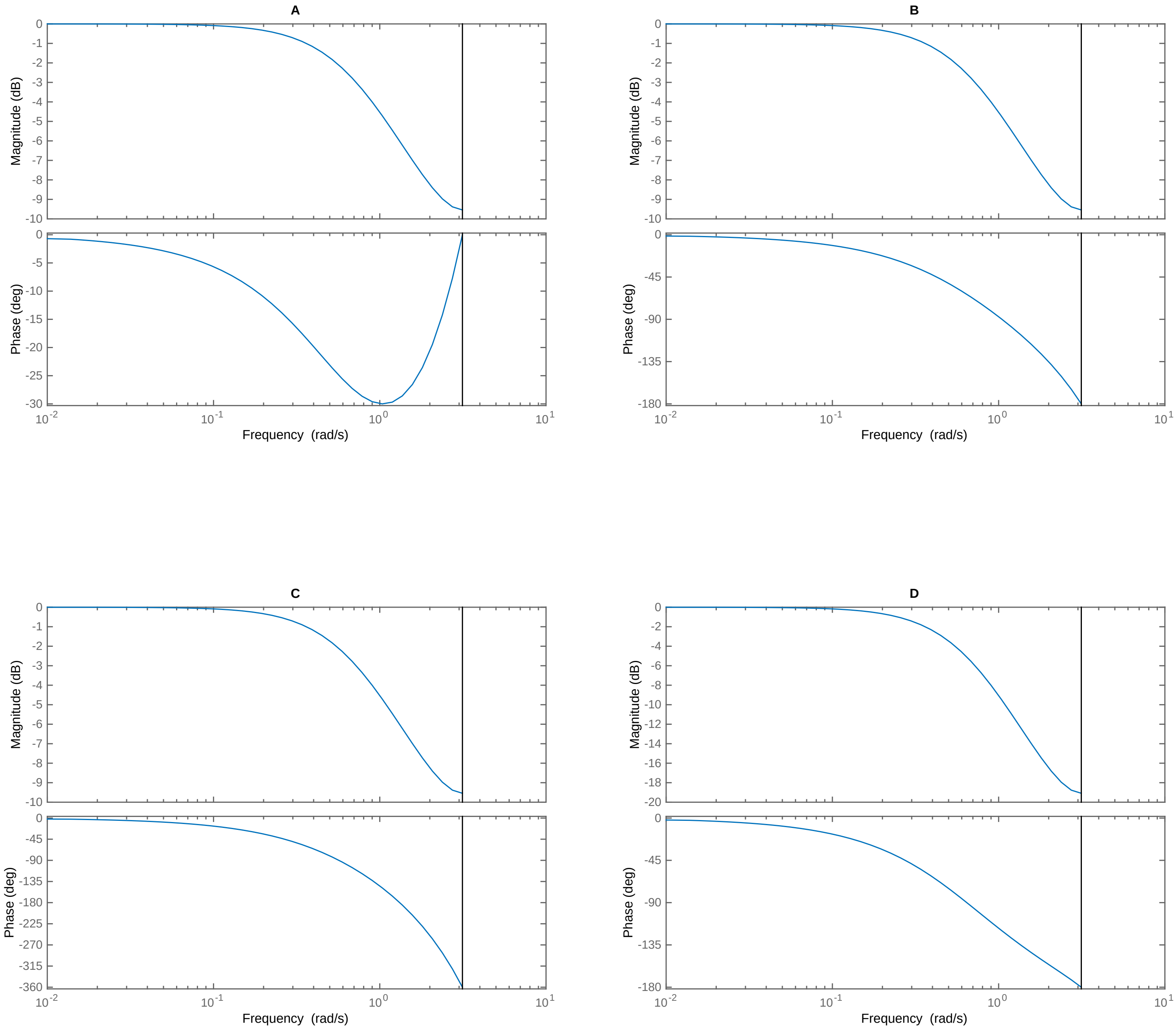


Fig. 1

3 Two independent random variables $X_1$ and $X_2$ have probability density functions (pdfs) $f_1(x)$ and $f_2(x)$.

(a) A new process $Y$ is formed from the sum of $X_1$ and $X_2$. Show that the conditional pdf of $Y$, given that $X_1 = x_1$, is given by

$$f(y|x_1) = f_2(y - x_1)$$

and show that the pdf of $Y$, $f_Y(y)$, is given by the convolution of $f_1$ with $f_2$. [25%]

(b) The characteristic function of a random variable $X$ is given by

$$\Phi_X(u) = E[e^{juX}]$$

where $E[.]$ is the expectation operator. For $Y = X_1 + X_2$, derive an expresson for $\Phi_Y(u)$ in terms of $\Phi_{X_1}(u)$ and $\Phi_{X_2}(u)$. [25%]

(c) Suppose there are $N$ mutually independent random variables $X_i$, $i = 1, \ldots, N$. Each $X_i$ has pdf $f_i(x)$ and characteristic function $\Phi_{X_i}(u)$. Derive an expression for $\Phi_Y(u)$, where $Y = \sum_{i=1}^{N} X_i$. [15%]

(d) A binary valued random variable $X$ has the Bernoulli distribution

$$P_X(x) = \begin{cases} 1 - p & x = 0 \\ p & x = 1 \end{cases}$$

for $0 < p < 1$. Derive an expression for the characteristic function $\Phi_X(u)$. [10%]

(e) An experimenter conducts $N$ independent tests, each of which succeeds with probability $p$ and fails with probability $1 - p$. The random variable $Y$ counts the number of successes, and has the binomial distribution

$$P_Y(n) = \binom{N}{n} p^n (1-p)^{N-n} \text{ for } n = 0, \ldots, N$$

Derive an expression for $\Phi_Y(u)$ and explain its relationship to the characteristic function of the Bernoulli distribution. [25%]

4 (a) For which of the following collections of lengths does there exist a binary prefix-free code? Justify your answer.

(i) $(1,2,3,4,5)$

(ii) $(1,2,3,3,4)$

(iii) $(2,2,2,2,2)$

(iv) $(2,2,3,3,3)$

[15%]

(b) A discrete random variable $X$ over the alphabet $\{A,B,C,D,E\}$ has the probability mass function

| $x$ | A | B | C | D | E |
|---|---|---|---|---|---|
| $P_X(x)$ | 0.8 | 0.07 | 0.06 | 0.02 | 0.05 |

(i) Calculate the entropy of $X$ in nats and in bits. [15%]

(ii) Design a Huffman code for $X$ and calculate the resulting expected codeword length. [15%]

(iii) Consider a discrete random process $X_1, X_2, X_3, \ldots$ consisting of independent and identically distributed random variables, each with the same probability mass function as $X$. State an upper bound on the expected codeword length per symbol attainable by Huffman coding for blocks of 1, 2 and 10 symbols. [20%]

(c) Let $l_1, l_2, l_3, l_4, l_5$ be the lengths of the codewords of the Huffman code for a random variable $X$ with probability mass function $(p_1, p_2, p_3, p_4, p_5)$ such that $p_1 \geq p_2 \geq p_3 \geq p_4 \geq p_5$.

(i) Show that $l_1 \leq l_2 \leq l_3 \leq l_4 \leq l_5$. [15%]

(ii) A random variable $X'$ has probability mass function

$$\left(p_1, p_2, p_3, p_4, \frac{p_5}{2}, \frac{p_5}{2}\right)$$

How do the lengths $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ of the Huffman code for $X'$ relate to the lengths $l_1, l_2, l_3, l_4, l_5$ of the Huffman code for $X$? [20%]

**END OF PAPER**

## Numerical Answers:

1. (d) $a = 1$, with stable poles at $z = 0.75 \pm i0.433$

(e) Stable system with steady state output $y_k = 1.5267\cos(k - 1.2997)$

2. (a) (i) C

(ii) Same plot with y axes rescaled by a factor of 10 (both magnitude and phase).

(b) (ii) $g(x) = tan(x)$.

4. (a) yes, no, no, yes

(b) (i) 0.76 nats, 1.10 bits

(ii) $E[L] = 1.4$ binary symbols per codeword

(iii) upper bounds 2.1, 1.6 and 1.2 for $K = 1, 2, 10$, respectively.