Version JS/4

EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 26 April 2016 9.30 to 11

Module 3F1

SIGNALS AND SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) If $\{g_k\}$ is the pulse response of a time-invariant linear discrete-time system, $\{u_k\}$ is an input sequence to this system, and $\{y_k\}$ is the resulting output sequence, state the relationship between $\{g_k\}$, $\{u_k\}$ and $\{y_k\}$ (assuming zero initial conditions). [10%]

(b) If $x_k * y_k$ denotes the convolution sum of two sequences $\{x_k\}$ and $\{y_k\}$, and $\mathcal{Z}\{x_k\}$ denotes the z-transform of a sequence $\{x_k\}$, show that

$$\mathcal{Z}\{x_k * y_k\} = \mathcal{Z}\{x_k\}\mathcal{Z}\{y_k\}$$

[30%]

(c) With notation as in part (a), a system's input-output behaviour is governed by the difference equation

$$y_k = u_{k-1} + 2u_{k-2} + u_{k-3}$$

Find the pulse response and the transfer function of this system. [20%]

(d) If the input sequence to the system in part (c) is given by

$$u_k = (-1)^k \quad \text{for } k \geq 0$$

find the resulting output sequence. [20%]

(e) Show that the output sequence you obtained in part (d) is consistent with the frequency response of the system defined in part (c). [20%]

2 (a) The continuous-time system with transfer function

$$G(s) = \frac{s^2 + \omega_o^2}{(s + 10\omega_o)^2}$$

acts as a perfect notch filter at frequency ω_o (namely, its frequency response is zero at this frequency). A discrete-time approximation with transfer function $H(z)$ is obtained by using the Euler transformation, namely

$$H(z) = G\left(\frac{z-1}{T}\right)$$

where T is the sampling period.

(i) Find the poles and zeros of $H(z)$. [15%]

(ii) Show that $H(z)$ is stable providing that

$$0 < T < \frac{1}{5\omega_o}$$

[15%]

(iii) It can be shown that, keeping ω_o fixed, $H(e^{j\omega_o T}) \to 0$ as $T \to 0$, so that the perfect notch-filter behaviour of $G(s)$ is approached as T is reduced. Show that the low-frequency and high-frequency behaviours of $H(z)$ also match those of $G(s)$ in the limit as T is reduced. [20%]

(b) (i) Define the terms *strict sense stationary* and *wide sense stationary* (WSS) as applied to random processes and state which implies the other. [20%]

(ii) A random process $X(t) = A\cos(\omega t + \phi)$ is defined by independent random variables ω and ϕ. The quantity ϕ is uniformly distributed over $[-\pi, \pi]$ and the random variable ω has density $f(\omega)$. Find $E[X(t)]$ and $r_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$ in terms of $f(\omega)$ and explain whether $X(t)$ is WSS. [30%]

3 (a) What is meant by the term *ergodic* as applied to random processes? [20%]

(b) Let $X(t)$ be a stationary process whose value at time t has a uniform probability density function (pdf) over the range $[-A, A]$. The autocorrelation function (ACF) of X has the form

$$r_{xx}(\tau) = Q_X e^{-2\lambda|\tau|}$$

for a constant λ. Determine Q_X in terms of A. [25%]

(c) Calculate $S_X(\omega)$, the power spectral density (PSD) of X. [25%]

(d) $X(t)$ is passed through a filter with impulse response

$$h(t) = \delta(t) - \delta(t - T)$$

The output of the filter is $Y(t)$. Calculate the ACF $r_{YY}(\tau)$ and the PSD $S_Y(\omega)$ for Y. [30%]

4 A study of a population of mice records the total number of offspring X in each litter and the number of offspring Y in the litter that are female. Assume that the total number of offspring is equally likely to be 1, 2 or 3, and that the probability that any particular individual offspring is female is $1/2$.

(a) Compute the entropy $H(X)$ of the total number of offspring. [10%]

(b) Compute the equivocation or conditional entropy $H(Y\,|\,X)$ of the number of female offspring given the total number of offspring. [15%]

(c) Compute the joint entropy of the total number of offspring and of the number of female offspring $H(X,Y)$. [10%]

(d) Design an optimal binary code for the pair (X,Y). [40%]

(e) The two numbers recorded by the study for each litter were initially held in an uncompressed database using two binary digits to record each number, for a total of $L_f = 4$ binary digits. What average compression factor $\kappa_1 = E[L]/L_f$ is achieved by your code, where L is the length of your representation for a database entry (X,Y). [10%]

(f) Compute the average compression factor $\kappa_2 = E[L_{SF}]/L_f$ attained by a Shannon-Fano code, where L_{SF} is the length of the Shannon-Fano representation for (X,Y). [10%]
Hint: you do not need to provide the actual code table for the Shannon-Fano code.

(g) What average compression factor per database entry κ_3 is achievable by encoding all database entries (X,Y) jointly using an arithmetic encoder, assuming a large number of records? [5%]

END OF PAPER

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