Version SJG/4

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 26 April 2017 9.30 to 11

Module 3F3

SIGNAL AND PATTERN PROCESSING

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) A source generates a stream of symbols S_n, $n = 0, 1, \ldots$ and each symbol takes one of two possible values, either A or B. The probability of symbol S_n depends only upon the value of symbol S_{n-1}. Let $p(i_n|i_{n-1})$ denote the probability that $S_n = i_n$ given $S_{n-1} = i_{n-1}$. These probabilities are given in the following table.

$p(i_n|i_{n-1})$

	$S_{n-1} = A$	$S_{n-1} = B$
$S_n = A$	0.7	0.1
$S_n = B$	0.3	0.9

(i) Let $p(i_0)$ denote the probability that $S_0 = i_0$. Explain how $p(i_n)$, the probability that $S_n = i_n$, may be calculated. [10%]

(ii) Show that a possible probability mass function for $p(i_n)$, for all $n \geq 0$, is [10%]

	$p(i_n)$
$i_n = A$	0.25
$i_n = B$	0.75

(iii) Show that the random process $S_0, S_2, S_4, \ldots$, generated by the same source but retaining only source symbols with even time indices, is a Markov chain, and determine its transition probability matrix. [30%]

(b) The characteristic function of a random variable X is defined using the mathematical expectation $\mathbb{E}$ as $\varphi_X(t) = \mathbb{E}[\exp(iXt)]$ where t is a real number.

(i) Let X have probability density function $f_X(x)$. Determine the relationship between $\varphi_X(t)$ and the Fourier transform of $f_X(x)$. [10%]

(ii) Let $f_X(x)$ be the following triangular shaped function

$$f_X(x) = 1/b\left(1 - \frac{|x|}{b}\right) \qquad \text{for } |x| \leq b$$

and $f_X(x) = 0$ for $|x| > b$. Determine $\varphi_X(t)$ (using the Data Book). [10%]

(iii) Express $\varphi_X(t)$ as a power series in t and hence find $\mathbb{E}[X^0]$, $\mathbb{E}[X^2]$ and $\mathbb{E}[X^4]$. [30%]

2 A stationary, ergodic random process $\{X_n\}$ is measured over a time interval $n = 0, 1, ..., N-1$, leading to a measured vector of samples:

$$\mathbf{x} = [x_0 \ x_1 \ ... \ x_{N-1}]^T .$$

It is possible to estimate an unknown quantity θ using the process $\{X_n\}$. Let $\hat{\theta}(\mathbf{x})$ denote the estimator of θ, which is a function of the measured data.

(a) Define the terms *bias* and *variance* for the estimator $\hat{\theta}$. [15%]

(b) The mean and autocorrelation function of the process are to be estimated according to the formulae:

$$\hat{\mu} = \frac{1}{N} \sum_{n=0}^{N-1} x_n$$

and

$$\hat{R}_{XX}[k] = \frac{1}{N-k} \sum_{n=0}^{N-1-k} x_n x_{n+k}, \ (k = 0, ..., N-1) .$$

Explain why these estimation formulae are valid, given the stated assumptions about the process. [10%]

(c) Determine whether each estimator in part (b) is unbiased. [30%]

(d) The mean value of the process is now assumed to be zero. Some autocorrelation function values are now estimated according to the above estimation formula, leading to:

$$\hat{R}_{XX}[0] = 10.5, \ \hat{R}_{XX}[1] = -9.1, \ \hat{R}_{XX}[2] = 7 .$$

It is required to predict the next value of the signal based upon previous values using a linear filter:

$$\hat{x}_{n+1} = h_0 x_n + h_1 x_{n-1}$$

Assuming that the estimated autocorrelation values are accurate, determine the coefficients h_0 and h_1 such that the mean-squared prediction error $\mathbb{E}[(\hat{x}_{n+1} - x_{n+1})^2]$ is minimised. [30%]

(e) Determine the mean-squared prediction error of this optimal filter and compare it with a filter which takes the previous value of the process as the prediction, i.e. $\hat{x}_{n+1} = x_n$, commenting on why this simpler estimator would not be expected to perform well. [15%]

3 A pilot tone in an RF communications channel is measured at the receiver in the following form:

$$X_n = A + B\sin(\omega n) + V_n$$

where $\{V_n\}$ is a white Gaussian noise process with zero mean and variance σ_V^2, $\omega < \pi$ is a known frequency of transmission, A is an unknown DC offset and B is an unknown received signal amplitude. It is required to estimate A and B from a measured vector of samples from the process $\{X_n\}$,

$$\mathbf{x} = [x_0\ x_1\ \ldots\ x_{N-1}]^T .$$

(a) For a particular set of parameter values $A = a$ and $B = b$, show that the total squared error term $\varepsilon = \sum_{n=0}^{N-1}(x_n - a - b\sin(\omega n))^2$ can be expressed in terms of the unknown parameter vector $\theta = \begin{bmatrix} a \\ b \end{bmatrix}$ as

$$\varepsilon = \mathbf{x}^T\mathbf{x} - 2\mathbf{x}^T\mathbf{G}\theta + \theta^T\mathbf{G}^T\mathbf{G}\theta$$

where $\mathbf{G}$ should be carefully defined. [20%]

(b) Show that the Maximum Likelihood (ML) estimator for the parameter vector can be found by minimising the following expression:

$$N\log(2\pi\sigma_V^2) + \frac{\varepsilon}{\sigma_V^2}$$

and hence that the ML estimator is

$$\theta^{\mathrm{ML}} = \mathbf{M}^{-1}\mathbf{b}$$

where

$$\mathbf{M} = \begin{bmatrix} N & \frac{\sin(N\omega/2)}{\sin(\omega/2)}\sin(\omega(N-1)/2) \\ \frac{\sin(N\omega/2)}{\sin(\omega/2)}\sin(\omega(N-1)/2) & N/2 - \frac{\sin(N\omega)}{2\sin(\omega)}\cos(\omega(N-1)) \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \sum_{n=0}^{N-1} x_n \\ \sum_{n=0}^{N-1} \sin(\omega n) x_n \end{bmatrix}.$$

You may find the following result helpful:

$$\sum_{n=0}^{N-1} \exp(inb) = \exp(i(N-1)b/2)\frac{\sin(Nb/2)}{\sin(b/2)}$$

[40%]

 (cont.

(c) With $\omega = \pi/5$ and $N = 1000$, show that the ML solution simplifies to:

$$\hat{a} = \frac{1}{N}\sum_{n=0}^{N-1} x_n, \qquad \hat{b} = \frac{2}{N}\sum_{n=0}^{N-1} \sin(\omega n) x_n \,. \qquad [20\%]$$

(d) Explain why this simplification occurs in terms of the columns of the matrix **G**. How should the data length be chosen relative to ω, to ensure that this is the case? [20%]

4 (a) For a random process X_n, $n = 0, 1, \ldots$ explain the difference between *strict-sense stationary* (SSS) and *wide-sense stationary* (WSS). Why might WSS be the more practical assumption for modelling of a real-world physical process? [20%]

(b) A zero-mean random process X_n has autocorrelation function $R_{XX}(k) = c$ for $k = 0$ and $R_{XX}(k) = 0$ for $|k| > 0$, where c is a constant. It is passed through a linear system with infinite impulse response $\{h_n\}_{n=-\infty}^{\infty}$. If Y_n denotes the output process, find an expression for the autocorrelation function $R_{YY}(k)$ of the system output in terms of the impulse response and c. [30%]

(c) Find $R_{YY}(k)$ when $h_n = 0$ for $n < 0$ and

$$h_n = a \exp(-nb)$$

for $n \geq 0$, where a, b are positive constants. [25%]

(d) Determine the power spectral density of the random process Y_n. [25%]

END OF PAPER