EGT2
ENGINEERING TRIPOS PART IIA

Monday 4 May 2015 2 to 3.30

Module 3M1

MATHEMATICAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3M1 data sheet (4 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Iterative methods can be used to compute approximate solutions to problems of the form $\boldsymbol{Ax} = \boldsymbol{b}$. List potential advantages and disadvantages of using iterative methods over direct methods. [20%]

(b) Consider a problem of the form

$$\boldsymbol{Ax} = \boldsymbol{x} \qquad (1)$$

where $\boldsymbol{A}$ is a Markov matrix. A Markov matrix is a square matrix with non-negative entries and the sum of each column vector of the matrix is equal to one.

(i) Show that a solution to (1) exists by showing that $\lambda = 1$ is an eigenvalue of $\boldsymbol{A}$.

Hint: consider the transposed problem $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{y}$ and recall that the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{A}^T$ are the same. [20%]

(ii) Prove that the largest absolute eigenvalue of $\boldsymbol{A}$ is equal to one. [30%]

(iii) Given that the largest eigenvalue of $\boldsymbol{A}$ is equal to one, describe an iterative algorithm to efficiently approximate $\boldsymbol{x}$ in (1). Apply the algorithm to find an approximation of $\boldsymbol{x}$ for the case

$$\boldsymbol{A} = \begin{bmatrix} 0.0 & 0.3 \\ 1.0 & 0.7 \end{bmatrix}$$

using three iterations and the starting vector $\boldsymbol{x}_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. [30%]

2 (a) Discuss briefly the differences between the penalty and barrier function methods of constrained optimization. [20%]

(b) An engineer is designing the tubes for a shell-and-tube heat exchanger. She has determined that the inner radius of each tube, r_i, should be 5 mm, and now has to choose the outer radius, r_o. She wants to minimize the mass per unit length of the tube, while ensuring that U, the effective heat transfer coefficient of the tube based on the inner radius, exceeds $2000\ \mathrm{W m^{-2} K^{-1}}$.

The tube is made out of copper of density ρ. The thermal conductivity of the copper is sufficiently high that the effective heat transfer coefficient of the tube is well approximated by

$$U = \frac{1}{r_i}\left[\frac{1}{r_i h_i} + \frac{1}{r_o h_o}\right]^{-1}$$

where h_i and h_o are the heat transfer coefficients on the inside and outside of the tube, respectively.

Formulate this task as a constrained optimization problem in standard form. [10%]

(c) Convert the problem into an equivalent unconstrained optimization problem through the use of a penalty function. [10%]

(d) Taking the initial value of the penalty parameter $p_1 = 0.001$, $\rho = 9000\ \mathrm{kg m^{-3}}$, $h_i = 3000\ \mathrm{W m^{-2} K^{-1}}$ and $h_o = 4000\ \mathrm{W m^{-2} K^{-1}}$, estimate using a Golden Section line search the value of r_o that minimizes the penalized objective. A suitable interval for r_o is between 6 mm and 9 mm. The search can be terminated when the interval has been reduced four times. The specified value of the penalty parameter p_1 assumes that U is evaluated in $\mathrm{W m^{-2} K^{-1}}$ and the mass per unit length in $\mathrm{kg\, m^{-1}}$. [40%]

(e) Complete the task of finding the optimal value of r_o, justifying the approach you choose to take. [20%]

3 A manufacturer of specialist equipment has entered into a contract to supply 40 units, 60 units and 50 units respectively in each of the next three months on an exclusive basis. The cost of producing x units in any given month is $(x^2 + 1000)$ k\$. The company can produce more units than required to meet supply in any given month and carry them forward to a subsequent month. However, it costs 20 k\$ per unit per month to store such units. The cost of hiring/firing employees to cope with fluctuating production from month to month is $10(x - y)^2$ k\$, where x is the production level in one month and y the production level in the subsequent month.

The managing director has asked you to identify the production levels for the next three months that minimize the total cost of production, storage and hiring/firing. There is no initial inventory of units in stock. Since the level of production needed after the present contract ends is unknown, you have been asked to assume that there will be no hiring/firing at the end of the contract, but you have been asked to account for the storage costs for a further month of any units produced in excess of the 150 needed to fulfil the contract.

(a) Show that, assuming the monthly production levels can be represented adequately by continuous real numbers, the optimization problem to be solved can be expressed in standard form as:

Minimize $f(x, y, z) = x^2 + y^2 + z^2 + 10(x - y)^2 + 10(y - z)^2 + 60x + 40y + 20z - C$

subject to

$$g_1(x, y, z) = 40 - x \le 0$$

$$g_2(x, y, z) = 100 - x - y \le 0$$

$$g_3(x, y, z) = 150 - x - y - z \le 0$$

where x, y and z are the levels of production in the first, second and third months of the contract respectively, and C is a constant. What is the value of C? [20%]

(b) Find the Kuhn-Tucker optimality conditions for this problem. [20%]

(c) Show that the case where monthly production exactly matches demand, i.e. $x = 40$, $y = 60$ and $z = 50$, does *not* represent a possible optimum. [10%]

(d) Given that at the optimum $\mu_1 = 0$ and $\mu_2 > 0$, where μ_1 and μ_2 are the Kuhn-Tucker multipliers associated with g_1 and g_2 respectively, find the monthly production levels that minimize $f(x, y, z)$ subject to the constraints. Comment on the practical implications of this solution and the associated values of the Kuhn-Tucker multipliers. [50%]

4 The transitions in a finite state-space Markov chain are governed by an $N \times N$ matrix, $\mathbf{P}$, with the following form

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{0} & \mathbf{P}_{22} \end{bmatrix}$$

where element i, j of the matrix $\mathbf{P}$ is the probability of transitioning to state j if the system is in state i. $\mathbf{P}_{11}$ is a square $m \times m$ matrix. The elements of $\mathbf{P}_{11}$, $\mathbf{P}_{12}$ and $\mathbf{P}_{22}$ are non-zero.

(a) What constraints must be satisfied by $\mathbf{P}$ for it to be a valid transition matrix for a finite state-space Markov chain? [10%]

(b) What equation must the *stationary distribution*, $\boldsymbol{\pi}$, satisfy? $\boldsymbol{\pi}$ is a row vector where

$$\boldsymbol{\pi} = \begin{bmatrix} \boldsymbol{\pi}_1 & \boldsymbol{\pi}_2 \end{bmatrix}$$

and $\boldsymbol{\pi}_1$ is an m-dimensional row vector. The simplest form of the equation should be given. Does the stationary distribution depend on the initial state? [20%]

(c) If all elements of $\mathbf{P}_{12}$ are zero, how does this change the answer to (b)? [15%]

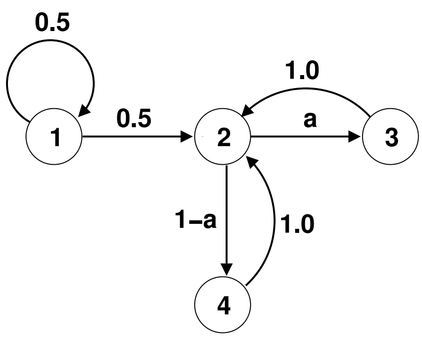

Fig. 1

(d) Figure 1 shows the state-diagram for a particular process.

(i) Write down the transition matrix for this process. [15%]

(ii) Find the stationary distribution for this process. [20%]

(iii) If the process starts in state 1, find an expression in terms of "a" for the expected number of transitions before the process is in state 4. What value of "a" results in this expected number of transitions being 4? [20%]

END OF PAPER

3M1 Mathematical Methods 2015

Answers

Q1 (b)(iii) $\boldsymbol{x}_3 = [\,0.447 \quad 1.553\,]^T$

Q2 (d) $I_5 = 7.146 \le r_o \le 7.584$ mm

(e) $r_o = 0.0075$ m or 7.5 mm

Q3 (d) $x = 49.808,\ y = 50.192,\ z = 50,\ \mu_2 = 35.769,\ \mu_3 = 116.154$

Q4 (b) $\pi_1 = \mathbf{0};\ \pi_2 = \pi_2 \mathbf{P}_{22}$

(d)(i) $\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & a & 1-a \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(d)(ii) $\pi = \begin{bmatrix} 0 & \frac{1}{2} & \frac{a}{2} & \frac{1-a}{2} \end{bmatrix}$

(d)(iii) $q_1 = \dfrac{3-a}{1-a};\ a = \frac{1}{3}$