EGT2
ENGINEERING TRIPOS PART IIA

Thursday 21 April 2016 2 to 3.30

Module 3M1

MATHEMATICAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachments: 3M1 data sheet (4 pages)
Figure for question 3, to be submitted with the solution
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Show that any real-valued matrix $\boldsymbol{M}$ admits the singular value decomposition

$$\boldsymbol{M} = \boldsymbol{U\Sigma V}^T$$

where the columns of $\boldsymbol{U}$ are the normalised eigenvectors of $\boldsymbol{MM}^T$, the columns of $\boldsymbol{V}$ are the normalised eigenvectors of $\boldsymbol{M}^T\boldsymbol{M}$, and the diagonal entries of $\boldsymbol{\Sigma}$ are the square roots of the non-zero eigenvalues of $\boldsymbol{MM}^T$. [40%]

(b) Indicate whether or not each of the following decompositions are valid singular value decompositions and explain why. For the valid cases, give the rank of the matrix.

$$\boldsymbol{M}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\boldsymbol{M}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{M}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{M}_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\boldsymbol{M}_5 = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & \sqrt{2/3} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

[20%]

(c) A particular fitting problem leads to the system $\boldsymbol{Mx} = \boldsymbol{b}$, where the singular value decomposition of $\boldsymbol{M}$ is given by:

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1.2 \times 10^{-9} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) Compute the condition number κ_2 of $\boldsymbol{M}$, and comment on the significance of the condition number for this matrix. [10%]

(ii) What is the effective rank of $\boldsymbol{M}$? [5%]

(iii) For $\boldsymbol{b}^T = \begin{bmatrix} 7 & 15 & 6 \end{bmatrix}$, find an approximate solution $\boldsymbol{x}$ that is insensitive to measurement noise in the vector $\boldsymbol{b}$. [25%]

2 A civil engineer is designing a water canal with a fixed cross-sectional area A. It must be designed so that its discharge capacity is maximised. The design variables (see Fig. 1) are the height h, the width of the base b and the angle of the sides ϕ. It can be shown that the discharge capacity is proportional to the inverse of the wetted perimeter p, which is given by

$$p = b + 2h \csc\phi$$

The cross-sectional area is easily shown to be given by

$$A = hb + h^2 \cot\phi$$

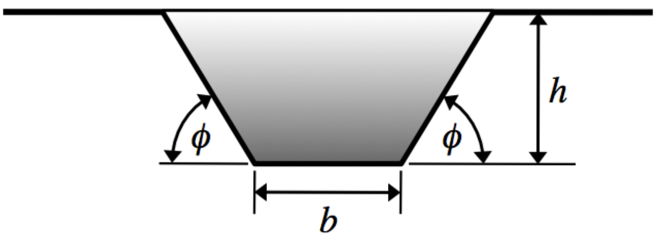


Fig. 1

(a) Formulate the design task as a constrained minimisation problem. [10%]

(b) By eliminating one of the design variables from the expression for the objective function using the equality constraint, show that the task can be formulated as an unconstrained (assuming non-negativity bounds on control variables can be neglected) minimisation problem:

$$\text{Minimise} \quad f = \frac{A}{h} - h\cot\phi + 2h\csc\phi$$ [10%]

(c) Taking $A = 100\ \text{m}^2$ and starting from an initial solution of $h = 5$ m and $\phi = \pi/4$ radians, complete one iteration of Newton's Method. [45%]

(d) By applying appropriate optimality criteria, find the optimal canal design for the $A = 100\ \text{m}^2$ case. Comment on the performance of Newton's Method observed in (c). [35%]

3 Figure 2 shows a schematic diagram in cross-section of the cone of a cone clutch.

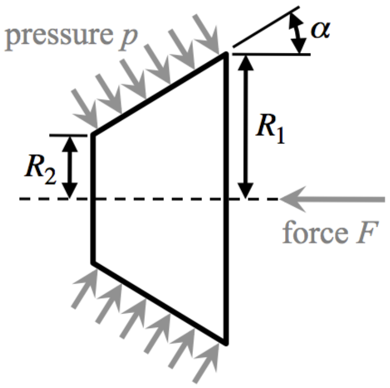


Fig. 2

In terms of the parameters defined in Fig. 2, the volume of the cone is given by

$$V = \tfrac{1}{3}\pi\cot\alpha\left(R_1^3 - R_2^3\right)$$

When the clutch is engaged there is a uniform pressure p acting between the cone and the cup (not shown). The pressure depends on the applied force F as follows:

$$p = \frac{F}{\pi\left(R_1^2 - R_2^2\right)}$$

The torque transmitted by the cone clutch is given by

$$T = \frac{2\pi f p}{3\sin\alpha}\left(R_1^3 - R_2^3\right)$$

where f is the coefficient of friction between the cone and cup.

(a) The volume V of the cone is to be minimised by varying R_1 and R_2 for the case where $\alpha = 30^\circ$, $f = 0.5$ and $F = 750\,\mathrm{N}$, subject to the following constraints:

g_1: $T \geq 15{,}000\,\mathrm{N\,cm}$	g_4: $R_1 \leq 40\,\mathrm{cm}$
g_2: $p \leq \dfrac{15}{4\pi}\,\mathrm{N\,cm^{-2}}$	g_5: $R_1 \geq 0\,\mathrm{cm}$
g_3: $R_2 \leq \frac{1}{2}R_1$	g_6: $R_2 \geq 0\,\mathrm{cm}$

Figure 3, a copy of which is provided as an attachment to this paper, shows various lines plotted in R_1-R_2 space. On this copy, indicate which line corresponds to each of the inequality constraints g_1 to g_6 and which line represents a contour of V. Full credit will only be obtained if choices are supported/justified by appropriate working/ reasoning. [40%]

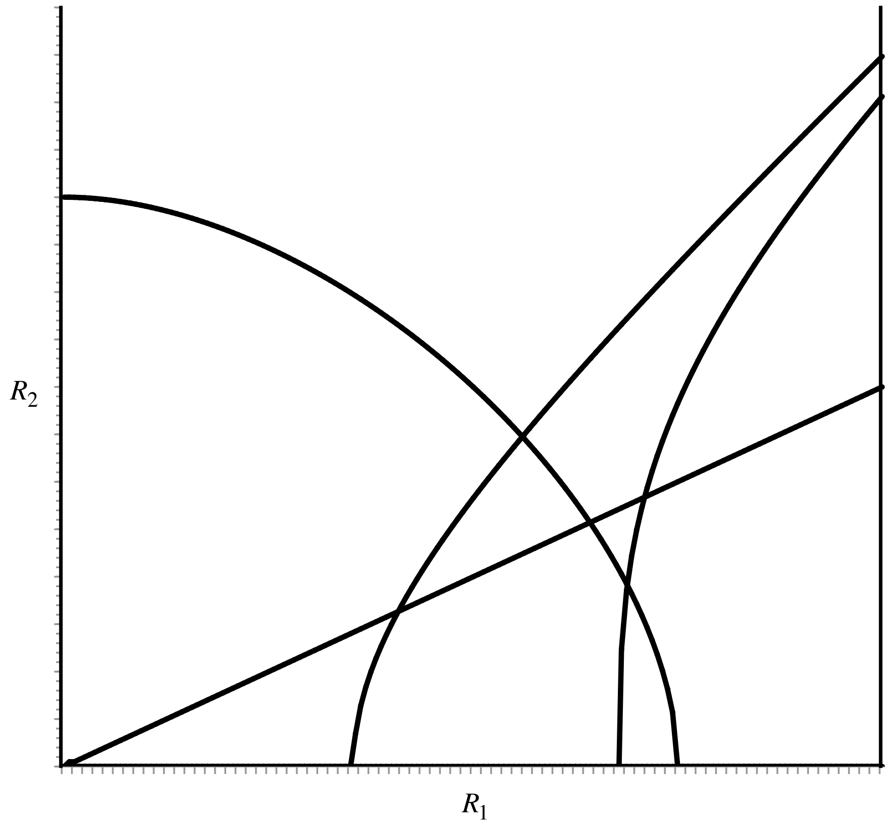


Fig. 3

(b) Identify the feasible region in your copy of Fig. 3, and hence, by considering contours of V, find and evaluate the minimum volume cone clutch design in this case. [25%]

(c) If constraint g_3 is to be eliminated, explain why it is not necessary to introduce an alternative constraint g_7 ($R_2 \leq R_1$) to ensure that R_2 is less than R_1. Identify the location in R_1-R_2 space of the optimum in this case. You are not required to find the corresponding values of R_1 and R_2. [15%]

(d) Find and evaluate the minimum volume cone clutch design if the torque is to be transmitted under *uniform wear* conditions, which require that

$$T = \frac{\pi f p R_2}{\sin \alpha}\left(R_1^2 - R_2^2\right)$$

rather than the expression for T given earlier. All the constraints specified except g_3 still apply. [20%]

4 A simple five-stage manufacturing process can be described by a Markov process. At each time instance an item is placed in State 1, the start of the process, just before each transition occurs. Completed items are then kept in State 5, the end of the process. Figure 4 shows the state-diagram for the process.

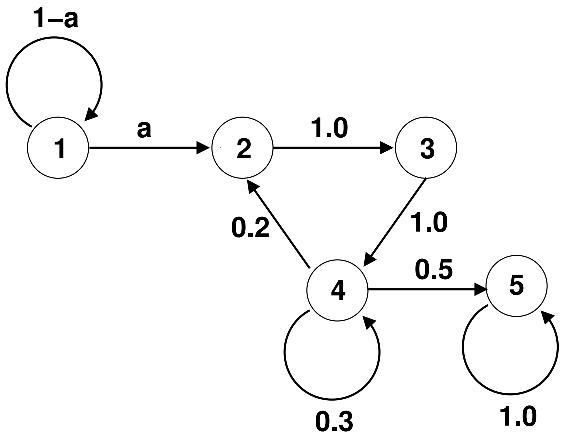


Fig. 4

(a) Write down the transition matrix, **P**, for this process. What is the stationary distribution for this process? [15%]

(b) What is the expected time for an item to first enter State 5? [30%]

(c) Write an expression in terms of **P** for the probability distribution over the states for a single item n steps after it entered the process. [10%]

(d) The process is run for N steps.

(i) Show that the expected number of items in each state can be expressed as

$$\pi\,(\mathbf{P}-\mathbf{A})\mathbf{B}^{-1}$$

What are π, **A** and **B**? [30%]

(ii) How does the value of "**a**" influence the expected number of items in each of the states? [15%]

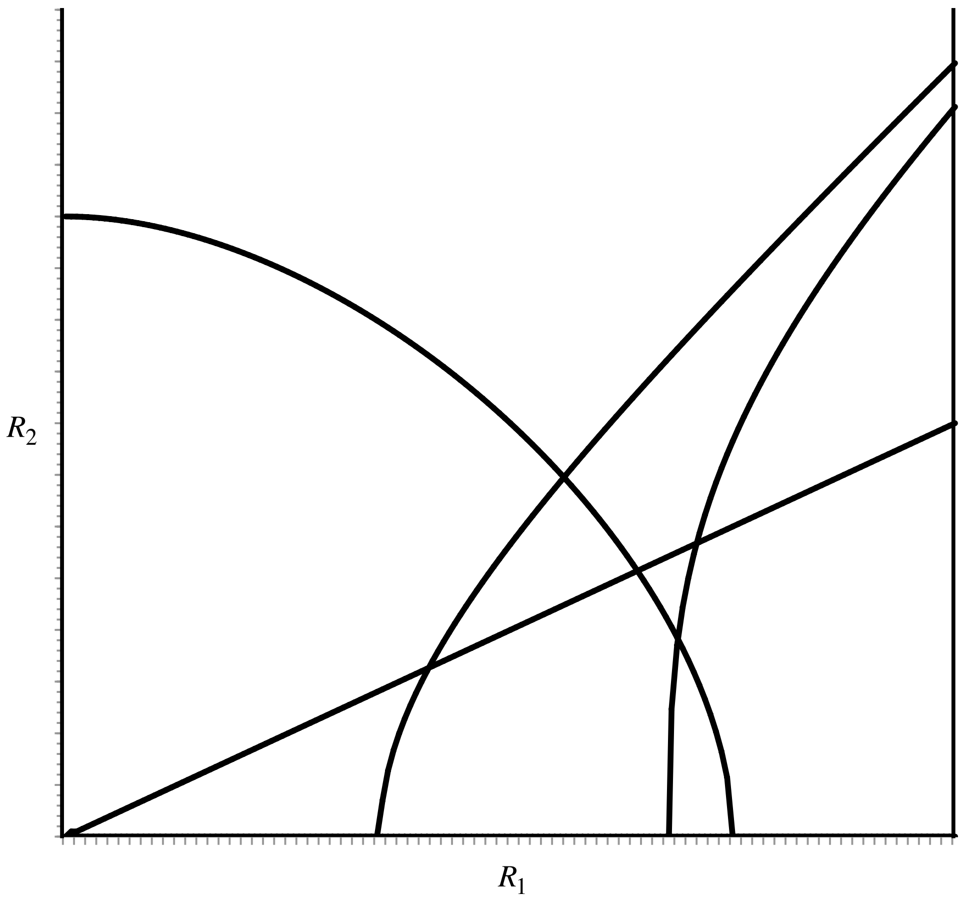
END OF PAPER

Version GTP/4

CANDIDATE No. ________________

ENGINEERING TRIPOS PART IIA

Thursday 21 April 2016, Module 3M1, Question 3



Copy of Fig. 3

(to be returned with your solution)

3M1 Mathematical Methods 2016

Answers

Q1 (b) $\boldsymbol{M}_1$: invalid because $\Sigma_2 < 0$; $\boldsymbol{M}_2$: valid, rank = 2; $\boldsymbol{M}_3$: valid, rank = 1; $\boldsymbol{M}_4$: invalid, $\boldsymbol{\Sigma}$ is not diagonal; $\boldsymbol{M}_5$: invalid, $\boldsymbol{U}$ is not orthogonal.

(c)(i) 7.5×10^9

(c)(ii) 2

(c)(iii) $\begin{bmatrix} \frac{7}{9\sqrt{2}} - \frac{5}{\sqrt{2}} \\ \frac{7}{9\sqrt{2}} + \frac{5}{\sqrt{2}} \\ 0 \end{bmatrix}$

Q2 (c) Solution after one iteration is $\begin{bmatrix} h \\ \phi \end{bmatrix} = \begin{bmatrix} 6.4812 \\ 1.0248 \end{bmatrix}$

(d) $\begin{bmatrix} h \\ \phi \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{100}{\sqrt{3}}} \\ \frac{\pi}{3} \end{bmatrix}$

Q3 (b) $R_1 = \frac{180}{7}$ cm, $R_2 \frac{180}{14}$ cm

(d) $R_2 = 20$ cm, $R_1 = \sqrt{600}$ cm

Q4 (a) $\mathbf{P} = \begin{bmatrix} 1-a & a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.2 & 0 & 0.3 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, $\boldsymbol{\pi} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ if $a > 0$

(b) $\frac{1}{a} + 4.8$

(c) $\boldsymbol{\pi}^{(n)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{P}^n$