EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 3 May 2017 9.30 to 11

Module 3M1

MATHEMATICAL METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachments: 3M1 data sheet (4 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Compute the l_1, l_2 and l_∞ norms for the following vectors:

$$\boldsymbol{u}^T = \begin{bmatrix}1 & 1 & 1 & 4\end{bmatrix},\ \boldsymbol{v}^T = \begin{bmatrix}-10 & 9 & 0 & 2\end{bmatrix},\ \boldsymbol{w}^T = \begin{bmatrix}-9 & 2 & 10 & 0\end{bmatrix}$$ [10%]

(b) (i) Compute the 1-norm, 2-norm and ∞-norm (operator norms) for the matrices:

$$\boldsymbol{A} = \begin{bmatrix}-3 & 0 & 0 & 0\\ 0 & 11 & 0 & 0\\ 0 & 0 & -12 & 0\\ 0 & 0 & 0 & 7\end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix}3 & -2\\ 4 & -1\end{bmatrix}$$ [20%]

(ii) Consider $n \times n$ matrices $\boldsymbol{C}$ and $\boldsymbol{D}$. If $\|\boldsymbol{C}\|_p = 5$ and $\|\boldsymbol{D}\|_p < 7$, where $p > 0$ is an integer, provide upper bounds for

$$\|\boldsymbol{C}+\boldsymbol{D}\|_p \quad \text{and} \quad \|\boldsymbol{CD}\|_p$$ [10%]

(c) A polynomial interpolation problem involves solving $\boldsymbol{Ap} = \boldsymbol{b}$ to find the coefficients of the polynomial, where $\boldsymbol{A}$ is the Vandermonde matrix:

$$\boldsymbol{A} = \begin{bmatrix}x_1^3 & x_1^2 & x_1 & 1\\ x_2^3 & x_2^2 & x_2 & 1\\ x_3^3 & x_3^2 & x_3 & 1\\ x_4^3 & x_4^2 & x_4 & 1\end{bmatrix}$$

For the case of interest, $x_1 = 800$, $x_2 = 801$, $x_3 = 802$ and $x_4 = 803$, and

$$\boldsymbol{A} = \boldsymbol{U}\begin{bmatrix}1.030\times10^9 & & & \\ & 1.792\times10^3 & & \\ & & 2.495\times10^{-3} & \\ & & & 2.606\times10^{-9}\end{bmatrix}\boldsymbol{V}^T$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthogonal matrices.

(i) Compute the condition number κ_2 for the case of interest and comment on its significance. [15%]

(ii) For an error $\delta\boldsymbol{b}$ in the right-hand-side vector, $\boldsymbol{b}$, show that the error in the solution, $\delta\boldsymbol{p}$, satisfies

$$\frac{\|\delta\boldsymbol{p}\|}{\|\boldsymbol{p}\|} \le \underbrace{\|\boldsymbol{A}\|\|\boldsymbol{A}^{-1}\|}_{\kappa(\boldsymbol{A})}\frac{\|\delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|}$$ [30%]

(iii) Suggest a simple reformulation of the particular interpolation problem considered in this question that would improve the conditioning of $\boldsymbol{A}$. [15%]

2 The annual power requirements of a large chemical processing plant are defined in terms of a function $E(x)$ that gives the total number of MW-hours required at power levels up to x MW. The company can meet its requirements by installing its own combined cycle gas turbine (CCGT) plant of capacity (in MW) x_1 for base load supply, its own diesel generators of total capacity (in MW) x_2 for meeting additional demand, and by purchasing electricity from the grid for satisfying peak demand only.

Let c_1 and c_2 be the yearly capital costs (per MW of capacity) of the CCGT plant and diesel generators, respectively, and r_1 and r_2 be their running costs (per MW-hour generated). Let e be the price of the electricity purchased from the grid (per MW-hour). The plant's peak power requirement is $P_{\max}$ (in MW).

(a) Show that, in order to find the balance of generating capacity that minimizes the cost of supplying electricity to the plant, an appropriate objective function is

$$f(x_1, x_2) = c_1 x_1 + c_2 x_2 + r_1 E(x_1) + r_2\left[E(x_1 + x_2) - E(x_1)\right] + e\left[E(P_{\max}) - E(x_1 + x_2)\right]$$

Identify any constraints that apply. [15%]

(b) Assuming that the optimal solution is interior to the constraints (i.e. that this can be treated as an unconstrained optimization problem),

(i) find the first-order necessary conditions governing the optimum, and

(ii) show that the second-order conditions require that

$$(r_1 - r_2)E''(x_1) > 0$$

and

$$(r_2 - e)E''(x_1 + x_2) > 0$$

where $E''(x) = \dfrac{d^2E(x)}{dx^2}$. [45%]

(c) Find the optimal balance of generating capacity if:

$$P_{\max} = 900 \text{ MW}$$

$$c_1 = £90000 \text{ MW}^{-1}$$

$$c_2 = £45000 \text{ MW}^{-1}$$

$$r_1 = £75 \text{ per MW-hour}$$

$$r_2 = £105 \text{ per MW-hour}$$

$$e = £150 \text{ per MW-hour}$$

$$E(x) = E_0 \sin\left(\frac{\pi x}{2P_{\max}}\right) \text{ with } E_0 = 3.942 \times 10^6 \text{ MW-hours}$$ [40%]

3 A gas company owns a pipeline network, sections of which are used to pump natural gas from its main field to its distribution centre. The network is shown in Fig. 1 below, where the direction of the arrows indicates the only direction in which the gas can be pumped. The numbers next to each network link indicate the capacity u_i in units of gas per month, and the cost c_i in $ per unit of using that link, in the format (u_i, c_i). The company currently produces 8000 units per month from its main field (node 1) and must transport all this gas to the distribution centre (node 5). The intermediate nodes (2, 3 and 4) are large pumping stations.

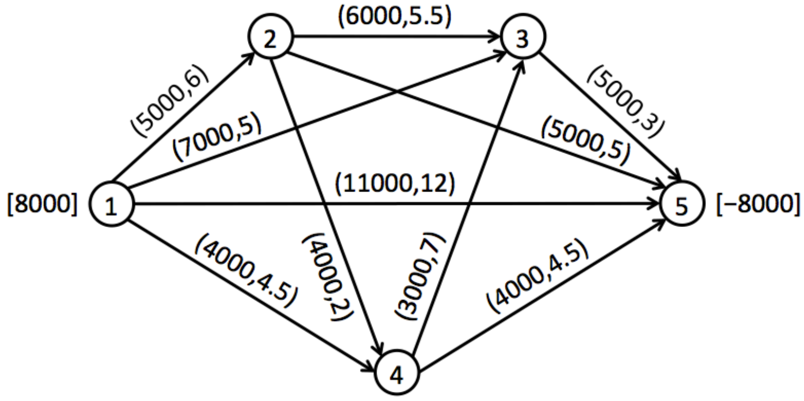

Fig. 1

The solution shown in Fig. 2 has been suggested. In this figure non-zero flows are shown next to the arc in question.

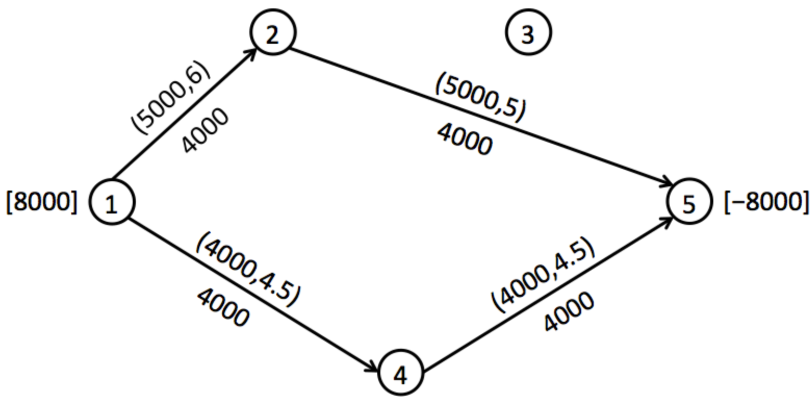

Fig. 2

(a) Explain why the solution as shown in Fig. 2 does not constitute a *spanning tree* of the network. How can it be modified so that it does constitute a basic feasible solution? [15%]

(b) Taking $y_1 = 0$, find the remaining simplex multipliers associated with a basic feasible solution for the flows shown in Fig. 2. Hence show that the solution suggested is not optimal.

You are reminded that, for such problems, the simplex multipliers for two nodes i and j are related by $c_{ij} - y_i + y_j = 0$, where c_{ij} is the cost per unit flow from i to j, and that the reduced costs for non-basic variables are given by $\bar{c}_{ij} = c_{ij} - y_i + y_j$. [30%]

(c) An alternative feasible basis is shown in Fig. 3. In this figure the solid arcs constitute the basis for the problem. The dashed arc 4–5 signifies a non-basic variable at its upper bound. All the arcs shown as grey dotted lines are non-basic variables at their lower bounds (0). Non-zero flows are shown next to the arc in question.

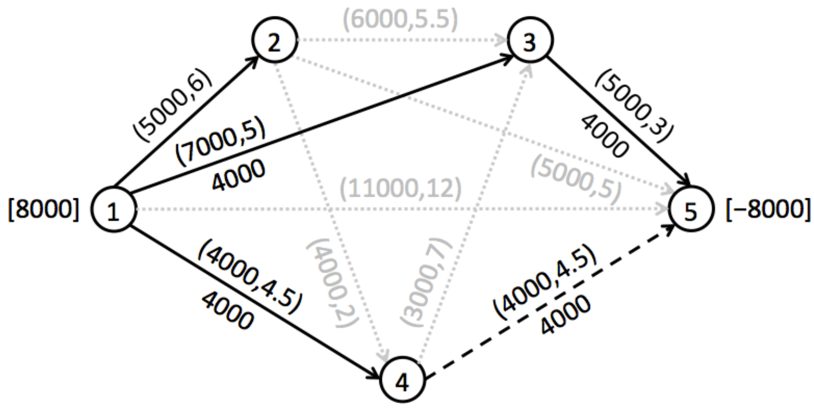

Fig. 3

Using the basis in Fig. 3 as a starting point, use the simplex network method to find the minimum cost network flow allocation. [55%]

4 The distribution of particles undergoing Brownian motion is a Wiener process with no drift. This process can be described by the following partial differential equation

$$\frac{\partial p(x,t)}{\partial t} = \alpha \frac{\partial^2 p(x,t)}{\partial x^2}$$

where x indicates the position and t indicates the time instance. α is a fixed scalar value.

(a) By using separation of variables, show that the general solution to this equation can be expressed as

$$p(x,t) = \int_{-\infty}^{\infty} a(k)\exp(-\alpha k^2 t)\exp(ikx)dk$$

where $i = \sqrt{-1}$ and $a(k)$ is a function of k. [25%]

(b) The initial condition for the process is set so that all particles start at the same position, $p(x,t) = \delta(x)$, where $\delta(x)$ is the Dirac delta function.

(i) Show that the solution with this boundary condition is Gaussian distributed and has the form

$$p(x,t) = \mathcal{N}(x \mid 0, b(t))$$

Find an expression for the variance $b(t)$. The following equality may be useful

$$\int_{-\infty}^{\infty} \exp(-\alpha k^2 t)\exp(ikx)dk = \sqrt{\frac{\pi}{\alpha t}}\exp\left(-\frac{x^2}{4\alpha t}\right)$$ [35%]

(ii) Describe with the aid of sketches how the distribution of particles changes over time, and the nature of the path of a single particle. [15%]

(c) A large number of particle trajectories are generated. What is the correlation between the trajectory positions at times t_1 and t_2, where $t_2 > t_1$? Note that the correlation between two variables x and y is defined as

$$\text{Corr}(x,y) = \frac{\mathcal{E}\left\{(x-\mu_x)(y-\mu_y)\right\}}{\sqrt{\mathcal{E}\left\{(x-\mu_x)^2\right\}\mathcal{E}\left\{(y-\mu_y)^2\right\}}}$$

where $\mathcal{E}\{\}$ is the expected value, μ_x is the mean of variable x and μ_y the mean of variable y. You should simplify your expression for the correlation as far as possible. [25%]

END OF PAPER

3M1 Mathematical Methods 2017

Answers

Q1 (a) $\|\boldsymbol{u}\|_1 = 7$; $\|\boldsymbol{u}\|_2 = \sqrt{19}$; $\|\boldsymbol{u}\|_\infty = 4$

$\|\boldsymbol{v}\|_1 = 21$; $\|\boldsymbol{v}\|_2 = \sqrt{185}$; $\|\boldsymbol{v}\|_\infty = 10$

$\|\boldsymbol{w}\|_1 = 21$; $\|\boldsymbol{w}\|_2 = \sqrt{185}$; $\|\boldsymbol{w}\|_\infty = 10$

(b)(i) $\|\boldsymbol{A}\|_1 = 12$; $\|\boldsymbol{A}\|_2 = 12$; $\|\boldsymbol{A}\|_\infty = 12$

$\|\boldsymbol{B}\|_1 = 7$; $\|\boldsymbol{B}\|_2 = \sqrt{15+10\sqrt{2}} \approx 5.398$; $\|\boldsymbol{B}\|_\infty = 5$

(c)(i) $\|\boldsymbol{C}+\boldsymbol{D}\|_p < 12$; $\|\boldsymbol{CD}\|_p < 35$

(c)(ii) 3.95×10^{17}

Q2 (b)(i) $\dfrac{\partial f}{\partial x_1} = c_1 + (r_1 - r_2)E'(x_1) + (r_2 - e)E'(x_1 + x_2) = 0$

$\dfrac{\partial f}{\partial x_2} = c_2 + (r_2 - e)E'(x_1 + x_2) = 0$

(c) $x_1 = 774.1\,\text{MW}$; $x_2 = 42.3\,\text{MW}$

Q3 (b) Arc 1–3, which is at its lower bound, has a negative reduced cost (–6.5), so the flow allocation specified cannot be optimal.

(c) The optimal flow allocation routes 5000 units from node 1 to node 5 via node 3 and 3000 units from node 1 to node 5 via node 4.

Q4 (b)(i) $b(t) = 2\alpha t$

(c) $\sqrt{t_1/t_2}$