EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 20 April 2016   9.30 to 11.00

Module 4A10

FLOW INSTABILITY

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4A10 Flow Instability data sheet (2 pages)
Charts of $St(Re)$, $C_D(Re)$, and $C_L(\overline{\alpha})$ & $C_D(\overline{\alpha})$
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Consider the flow of two incompressible inviscid fluids in horizontal parallel infinite streams of different velocities and densities, one stream above the other. In the undisturbed base state, a horizontal interface at $z = 0$ separates the upper stream $(z > 0)$, of uniform density ρ_2 and velocity $U_2 > 0$, from the lower stream $(z < 0)$, of uniform density ρ_1 and velocity $U_1 > 0$. The pressure $P(z)$ in the fluids is hydrostatic with a pressure p_0 at the interface.

(a) Explain why the governing equation for the flow is the Laplace equation for the velocity potential ϕ. [10%]

(b) Develop, with reasoning, the appropriate boundary conditions for the flow. Use ϕ_1 and ϕ_2 to denote the velocity potential for the lower and upper streams, respectively. [30%]

(c) With t denoting time and x denoting the horizontal coordinate, consider small amplitude disturbances to the interface $z = \eta(x,t)$ and to the velocity potentials of the form:

$$\eta(x,t) = \hat{\eta} e^{ikx+st}$$

$$\phi_1'(x,z,t) = \hat{\phi}_1(z) e^{ikx+st}$$

$$\phi_2'(x,z,t) = \hat{\phi}_2(z) e^{ikx+st}$$

for wavenumber k, growth rate s and constant $\hat{\eta}$.

Perform a linear stability analysis to show that

$$\rho_1(kg + (s + ikU_1)^2) = \rho_2(kg - (s + ikU_2)^2)$$

where g is the acceleration due to gravity. [60%]

2 A slender planar jet of liquid, bounded by free plane surfaces at $z = \pm a$, is moving at a steady velocity in still air. The liquid is inviscid, of density ρ and with surface tension γ, and the flow is incompressible.

(a) Use dimensional arguments to deduce a scaling for the growth rate s of a disturbance on the liquid-air interface. [15%]

(b) Explain the principles and approach behind a linear stability analysis. Include a description of why a normal mode analysis may be regarded as enabling the influence of all possible small amplitude disturbances to the flow to be assessed. [20%]

(c) Using a linear stability analysis, deduce an expression for the growth rate s of small amplitude disturbances on the liquid-air interface and, hence, assess the stability of the jet. [65%]

Fig. 1

3 Figure 1 shows frames from a movie taken in freezing weather on a UK motorway in January 2015. The lampposts are oscillating from side to side at 1.2 Hz with a tip deflection of ± 1 m. For this question, three charts are provided. The first plots Strouhal number, fD/U, as a function of Reynolds number, Re, for the frequency, f, in Hertz of vortex shedding around a round cylinder of diameter, D, in a wind of speed U. The second plots drag coefficient, C_D, as a function of Reynolds number, Re, for a round cylinder. The third plots lift coefficient, C_L, and drag coefficient, C_D, as a function of angle $\overline{\alpha}$ for a high Reynolds number flow around a cylinder that features a small ridge at angle $\overline{\alpha}$ to the flow direction.

(a) Three physical mechanisms have been proposed for this oscillation:

(i) vortex shedding,

(ii) changes in the drag coefficient with Reynolds number,

(iii) galloping.

Referring to the movie frames and charts as required, but without any calculations, explain how each of these mechanisms could cause the lampposts to oscillate. [60%]

(b) Using reasonable estimates for the wind speed, lamppost diameter, and air temperature, deduce which of these mechanisms is responsible for this oscillation. Show all calculations and all reasoning. [40%]

4 The Navier–Stokes equation for an incompressible flow in three spatial dimensions is:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \nabla^2 \mathbf{u} = -\nabla \frac{p}{\rho} \ .$$

For shear flow instabilities, this equation is often modelled by the Ginzburg-Landau equation in one spatial dimension:

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} - \nu \frac{\partial^2 \phi}{\partial x^2} = \alpha \phi \ ,$$

where ϕ represents a flow variable, such as perturbation velocity. In this question, the parameters U, ν, and α are independent of x and t.

(a) In the Ginzburg-Landau equation, what do U, ν, and α represent physically? [20%]

(b) By substituting $\phi = \phi_0 \mathrm{e}^{\mathrm{i}(kx - \omega t)}$ into the Ginzburg-Landau equation, derive the relationship between the complex frequency, ω, and the complex wavenumber, k. [10%]

(c) For given parameters (U, ν, α), sketch the temporal growth rate as a function of k, for real k. [10%]

(d) Under what conditions on U, ν, and α is the flow unstable? [10%]

(e) Find the absolute complex wavenumber, k_0, and absolute complex frequency, ω_0, for which the group velocity is zero. In the (U, α) space, sketch regions of stable flow, convectively unstable flow, and absolutely unstable flow. Bearing in mind your answer to (a), what physical insight does this give about the influence of U, ν, and α on shear flow instabilities? [50%]

END OF PAPER

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EQUATIONS OF MOTION

For an incompressible isothermal viscous fluid:

Continuity $\nabla \cdot \boldsymbol{u} = 0$

Navier Stokes $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

IRROTATIONAL FLOW $\nabla \times \boldsymbol{u} = 0$

velocity potential ϕ,

$$\boldsymbol{u} = \nabla \phi \text{ and } \nabla^2 \phi = 0$$

Bernoulli's equation

for inviscid flow $\dfrac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} =$ constant throughout flow field.

KINEMATIC CONDITION AT A MATERIAL INTERFACE

A surface $z = \eta(x,y,t)$ moves with fluid of velocity $\boldsymbol{u} = (u, v, w)$ if

$$w = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta \quad \text{on } z = \eta(x,t).$$

For η small and $\boldsymbol{u}$ linearly disturbed from $(U,0,0)$

$$w = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad \text{on } z = 0.$$

SURFACE TENSION σ AT A LIQUID-AIR INTERFACE

Potential energy

The potential energy of a surface of area A is σA.

Pressure difference

The difference in pressure Δp across a liquid-air surface with principal radii of curvature R_1 and R_2 is

$$\Delta p = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

For a surface which is almost a circular cylinder with axis in the x-direction, $r = a + \eta(x, \theta, t)$ (η is very small so that η^2 is negligible)

$$\Delta p = \frac{\sigma}{a} + \sigma \left(-\frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 \eta}{\partial \theta^2} \right),$$

where Δp is the difference between the internal and the external surface pressure.

For a surface which is almost plane with $z = \eta(x,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = -\sigma \frac{\partial^2 \eta}{\partial x^2}$$

where Δp is the difference between pressure at $z = \eta^+$ and $z = \eta^-$.

ROTATING FLOW

In steady flows with circular streamlines in which the fluid velocity and pressure are functions of radius r only:

Rayleigh's criterion

The flow is unstable / stable to inviscid axisymmetric disturbances if Γ^2 decreases / increases with r.

$\Gamma = 2\pi r V(r)$ is the circulation around a circle of radius r.

Navier Stokes equation simplifies to

$$0 = \mu \left(\frac{d^2 V}{dr^2} + \frac{1}{r} \frac{dV}{dr} - \frac{V}{r^2} \right)$$

$$-\rho \frac{V^2}{r} = -\frac{dp}{dr}.$$

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STABILITY OF PARALLEL SHEAR FLOW

Rayleigh's inflexion point theorem

A parallel shear flow with profile $U(z)$ is only unstable to inviscid perturbations if

$$\frac{d^2U}{dz^2}=0 \quad \text{for some } z.$$

CONVECTIVE FLOW

The Boussinesq approximation leads to

$$\nabla\cdot\boldsymbol{u}=0$$

$$\frac{D\boldsymbol{u}}{Dt}=-\frac{1}{\rho_0}\nabla p+(1-\alpha(T-T_0))\boldsymbol{g}+\upsilon\nabla^2\boldsymbol{u}$$

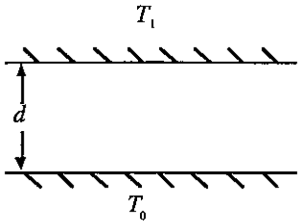
and

$$\frac{DT}{Dt}=\kappa\nabla^2 T$$

Rayleigh-Bénard convection

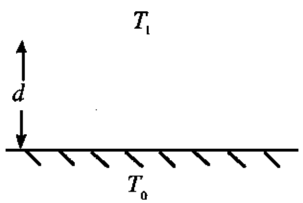
A fluid between two **rigid** plates is unstable when

$$\text{Ra}\geq 1708$$



A liquid with a **free** upper surface is unstable when

$$\frac{\text{Ra}}{\text{Ra}_c}+\frac{\text{Ma}}{\text{Ma}_c}\geq 1$$



where

$$\text{Ra}=\frac{g\alpha(T_0-T_1)d^3}{\upsilon\kappa},\quad \text{Ma}=\frac{\chi(T_0-T_1)d}{\rho\upsilon\kappa}\quad \text{with } \chi=-\frac{d\sigma}{dT}$$

$\text{Ra}_c\approx 670$ $\quad$ $\text{Ma}_c\approx 80.$

USEFUL MATHEMATICAL FORMULA

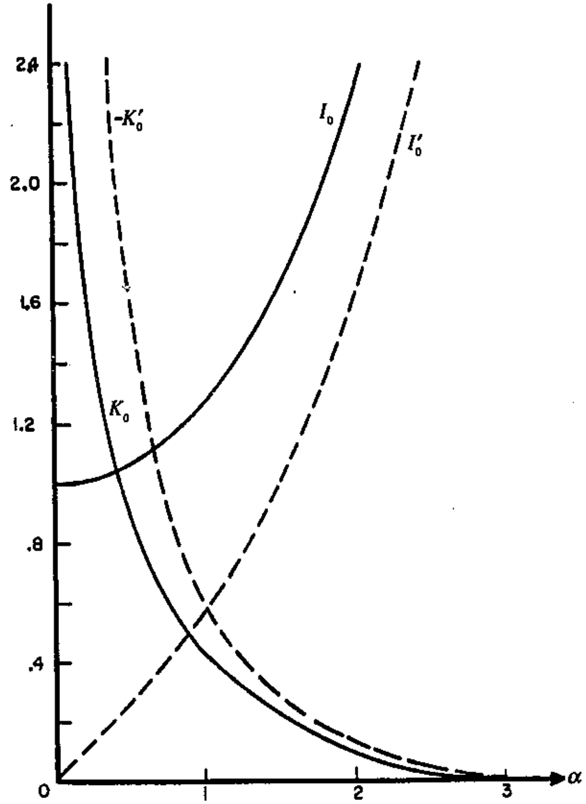
Modified Bessel equation

$I_0(kr)$ and $K_0(kr)$ are two independent solutions of

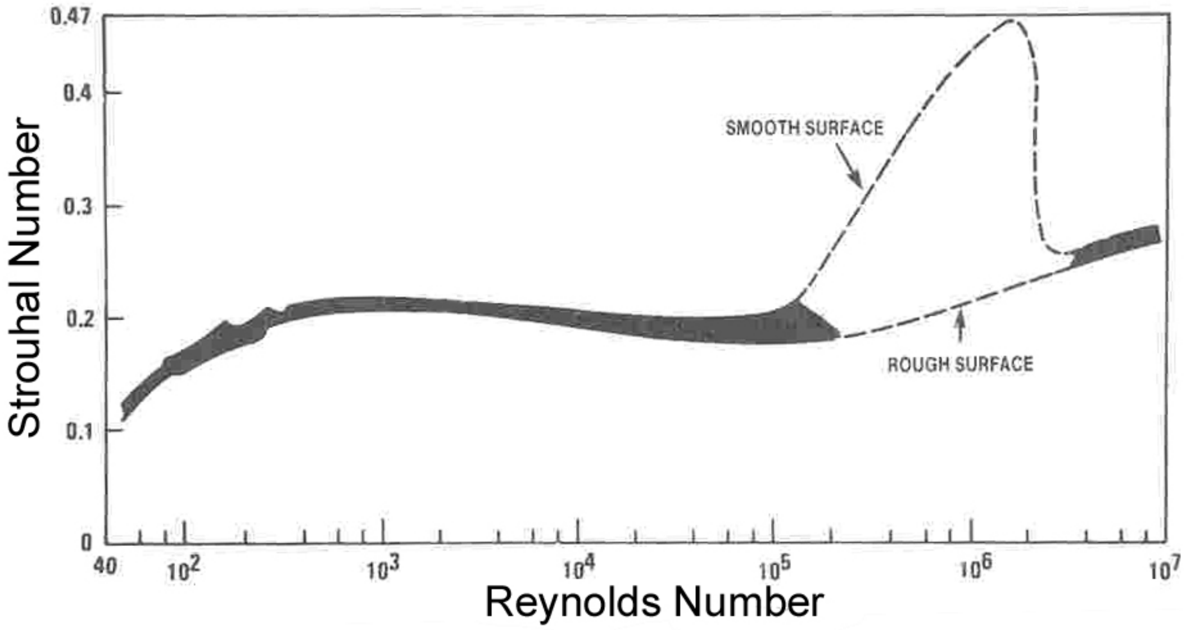
$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-k^2f=0.$$

$I_0(kr)$ is finite at $r=0$ and tends to infinity as $r\to\infty$,

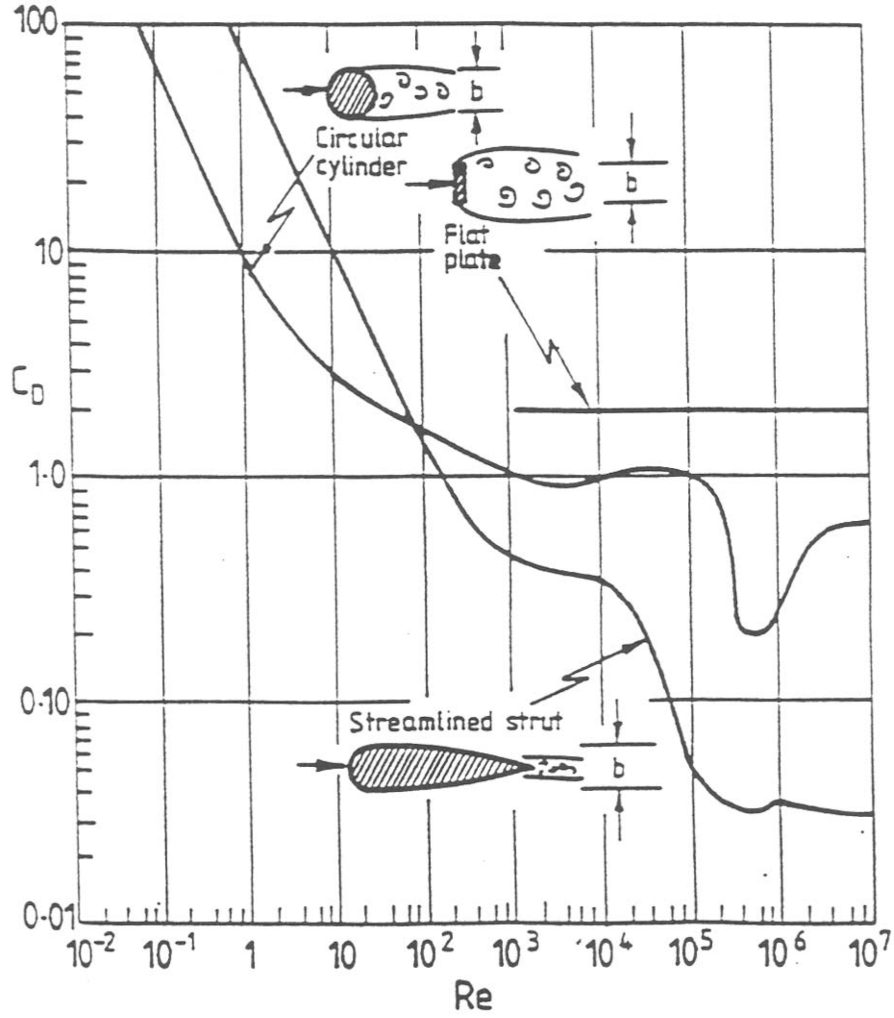
$K_0(kr)$ is infinite at $r=0$ and tends to zero as $r\to\infty$.



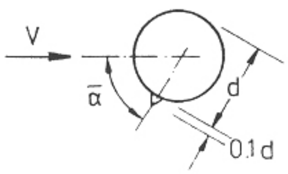
$I_0(\alpha)$, $K_0(\alpha)$, $I_0'(\alpha)$, $K_0'(\alpha)$
where $'$ denotes a derivative

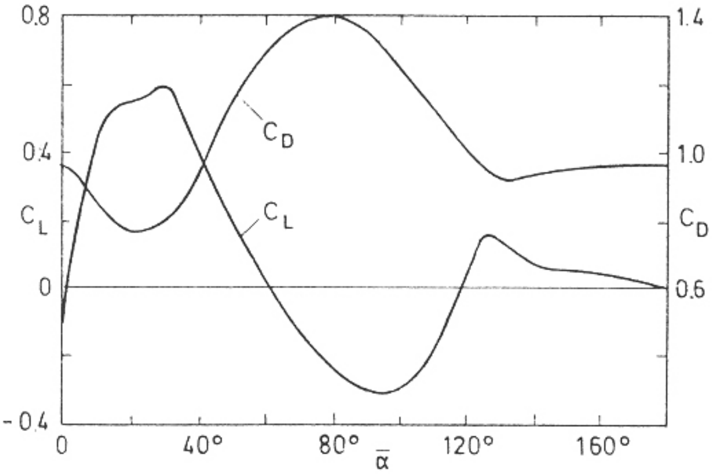


Strouhal Number
0.47
0.4
0.3
0.2
0.1
0
SMOOTH SURFACE
ROUGH SURFACE
40
10^2
10^3
10^4
10^5
10^6
10^7
Reynolds Number



100
10
1·0
0·10
0·01
C_D
Circular cylinder
Flat plate
Streamlined strut
b
10^{-2} 10^{-1} 1 10 10^2 10^3 10^4 10^5 10^6 10^7
Re


V
α
d
0.1d


0.8
0.4
C_L
0
-0.4
1.4
1.0
C_D
0.6
C_D
C_L
0
40°
80°
120°
160°
α