EGT3
ENGINEERING TRIPOS PART IIB

Friday 27 April 2018 9.30 to 11.10

Module 4A10

FLOW INSTABILITY

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book
Attachment: 4A10 Flow Instability data sheet (2 pages).

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 The centrifugal instability is analysed theoretically by letting a liquid of viscosity ν occupy the gap between two infinitely long concentric circular cylinders. The inner cylinder has radius r_1 and angular velocity Ω_1, and the outer cylinder has radius $r_2 > r_1$ and angular velocity Ω_2. The width d of the gap is narrow so that $d = r_2 - r_1 \ll r_1$. The cylinders rotate in the same sense ($\Omega_1 > 0$, $\Omega_2 > 0$) and at similar velocities ($\Omega_1 \approx \Omega_2$). The vertical axis of each cylinder is aligned with the coordinate z. The radial coordinate is denoted r. The dimensionless coordinate x, defined as $x = (r - r_1)/d$, spans the gap such that the liquid is confined by cylinder walls at $x = 0$ and $x = 1$.

To investigate the stability of the steady, purely rotary, base flow it is subjected to infinitesimal axisymmetric disturbances. For the velocity perturbation in the radial direction u'_r, normal mode solutions are sought of the form

$$u'_r = \hat{u}_r(r)\cos(nz)e^{st}$$

where n denotes the vertical wavenumber and s is the growth rate of the disturbance. You may assume that the radial velocity $\hat{u}_r(r)$ is governed by the following differential equation

$$\left[\left(\frac{d^2}{dx^2} - (nd)^2\right) - \frac{sd^2}{\nu}\right]^2 \left(\frac{d^2}{dx^2} - (nd)^2\right)\hat{u}_r = -(nd)^2 T \hat{u}_r \qquad (1)$$

where the Taylor number is

$$T = \frac{2(\Omega_1 {r_1}^2 - \Omega_2 {r_2}^2)d^3}{\nu^2 r_1}\left(\frac{\Omega_1 + \Omega_2}{2}\right)$$

(a) Using N to denote the radial wavenumber of the marginal state, solve Equation (1) subject to the boundary conditions

$$\hat{u}_r = \frac{d^2\hat{u}_r}{dx^2} = \frac{d^4\hat{u}_r}{dx^4} \quad \text{at} \quad x = 0 \quad \text{and} \quad x = 1$$

to show that the condition for marginal stability may be expressed as

$$T = \frac{1}{(nd)^2}\left[N^2\pi^2 + (nd)^2\right]^3$$

[45%]

(b) Derive the minimum value of the Taylor number that gives rise to marginal stability under the boundary conditions given in part (a) and comment on the implications for the vertical structure of the flow. [35%]

(c) Indicate on a clearly annotated sketch of Taylor number against dimensionless vertical wavenumber the regions of parameter space that correspond to stable and unstable flow. [20%]

2 Consider the two-dimensional, inviscid and incompressible flow of two homogeneous layers. The upper layer $(z > 0)$ has a density ρ_1 and flows with a uniform horizontal velocity U_1 in the x-direction. The lower layer $(z < 0)$ has a density $\rho_2 > \rho_1$ and flows with a uniform horizontal velocity U_2. In the base state, a horizontal interface at $z = 0$ separates the layers and pressure p is

$$p = \begin{cases} p_0 - \rho_1 g z & z > 0 \\ p_0 - \rho_2 g z & z < 0 \end{cases}$$

where $p_0 = p(z = 0)$ and g is the acceleration due to gravity.

(a) By considering small amplitude disturbances to the interface in time t and space x of the form

$$\eta(x,t) = \hat{\eta} e^{ikx+st}$$

for horizontal wavenumber k and growth rate s, investigate the temporal stability of the base flow to show that the dispersion relationship is

$$\rho_1 \left[kg - (s + U_1 ik)^2\right] = \rho_2 \left[kg + (s + U_2 ik)^2\right]$$

[75%]

(b) Discuss the implications for the stability of this two-layer system to small amplitude disturbances as the layer velocities and layer densities are varied. [25%]

3 A horizontal strut on a structure in a tidal river has square cross-section with side length $d = 0.24\,\text{m}$ and mass per unit length $m_a = 70\,\text{kg}\,\text{m}^{-1}$. The natural frequency is $\omega_a = 315\,\text{rad}\,\text{s}^{-1}$ when not submerged, and the damping factor $\zeta = 0.01$ when submerged. The damping factor is defined such that the equation of motion for the displacement y of an unforced mass-spring-damper system is $m\ddot{y} + 2m\zeta\omega\dot{y} + ky = 0$, where m is the notional mass, ω is the frequency and k is the stiffness.

The following information may be useful in this question: the density of seawater is $1025\,\text{kg}\,\text{m}^{-3}$; the added mass per unit length for a square cross-section is $1.51\rho\pi d^2$ where ρ denotes density; and for small angles of attack, α (in radians), the vertical force coefficient c_y for this shape is 2α.

(a) The tide comes in and water at speed U completely submerges the strut. Calculate the speed at which the strut will start oscillating. Explain physically what causes this oscillation. [50%]

(b) To prevent these oscillations, an engineer proposes to attach a mass to the strut using a spring and a damper. Explain physically why this might reduce the oscillation amplitude. If the spring stiffness is to be $2 \times 10^4\,\text{N}\,\text{m}^{-1}$, what mass should be used? What should be taken into account if this mass is also submerged? [30%]

(c) Propose better ways to damp or prevent the oscillations of the strut when it is submerged. [20%]

4 Figure 1 is a snapshot of a vertical round helium jet discharging into still air. Figure 2 is the power spectral density of the signal from a hot wire anemometer placed one jet radius downstream of the exit plane along the centreline.

(a) By interpreting Figs 1 and 2, describe the motion of the helium jet. [10%]

(b) Using the concepts of absolute and convective instability, explain how this motion arises. [20%]

(c) A local stability analysis is performed on the flow at the streamwise location of the hot wire anemometer. It assumes that perturbations have the form $e^{i(kx-\omega t)}$, where k is the complex wavenumber and ω is the complex angular frequency. The coordinate x is measured along the centreline of the jet with origin at the exit plane and t is time. Give the physical meaning of the real and imaginary components of k and ω. Sketch contours of ω_i in the complex $k-$plane, where ω_i is the complex part of ω; indicate the contour or contours that have $\omega_i = 0$; and highlight any other relevant features. [50%]

(d) Explain why a spatial stability analysis of the flow at this streamwise location has no physical meaning. [20%]

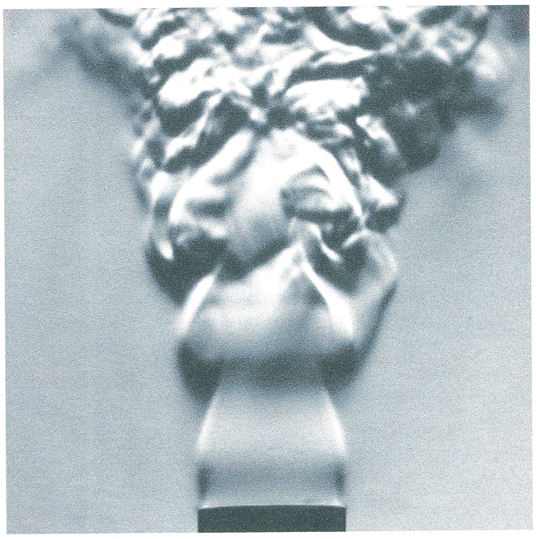

Fig. 1

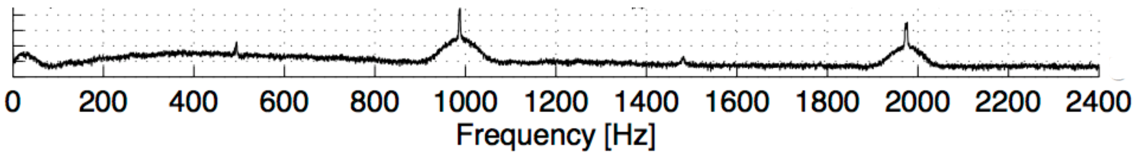

Fig. 2

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Module 4A10 Data Card

EQUATIONS OF MOTION

For an incompressible isothermal viscous fluid:

Continuity $\nabla \cdot \boldsymbol{u} = 0$

Navier Stokes $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

IRROTATIONAL FLOW $\nabla \times \boldsymbol{u} = \boldsymbol{0}$

velocity potential ϕ,

$$\boldsymbol{u} = \nabla \phi \text{ and } \nabla^2 \phi = 0$$

Bernoulli's equation

for inviscid flow $\dfrac{p}{\rho} + \tfrac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} =$ constant throughout flow field.

KINEMATIC CONDITION AT A MATERIAL INTERFACE

A surface $z = \eta(x,y,t)$ moves with fluid of velocity $\boldsymbol{u} = (u, v, w)$ if

$$w = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta \quad \text{on } z = \eta(x,t).$$

For η small and $\boldsymbol{u}$ linearly disturbed from $(U,0,0)$

$$w = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad \text{on } z = 0.$$

SURFACE TENSION σ AT A LIQUID-AIR INTERFACE

Potential energy

The potential energy of a surface of area A is σA.

Pressure difference

The difference in pressure Δp across a liquid-air surface with principal radii of curvature R_1 and R_2 is

$$\Delta p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

For a surface which is almost a circular cylinder with axis in the x-direction, $r = a + \eta(x, \theta, t)$ (η is very small so that η^2 is negligible)

$$\Delta p = \frac{\sigma}{a} + \sigma\left(-\frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2 \eta}{\partial \theta^2}\right),$$

where Δp is the difference between the internal and the external surface pressure.

For a surface which is almost plane with $z = \eta(x,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = -\sigma \frac{\partial^2 \eta}{\partial x^2}$$

where Δp is the difference between pressure at $z = \eta^+$ and $z = \eta^-$.

ROTATING FLOW

In steady flows with circular streamlines in which the fluid velocity and pressure are functions of radius r only:

Rayleigh's criterion

The flow is unstable / stable to inviscid axisymmetric disturbances if Γ^2 decreases / increases with r.

$\Gamma = 2\pi r V(r)$ is the circulation around a circle of radius r.

Navier Stokes equation simplifies to

$$0 = \mu\left(\frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2}\right)$$

$$-\rho \frac{V^2}{r} = -\frac{dp}{dr}.$$

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STABILITY OF PARALLEL SHEAR FLOW

Rayleigh's inflexion point theorem

A parallel shear flow with profile $U(z)$ is only unstable to inviscid perturbations if

$$\frac{d^2U}{dz^2}=0 \quad \text{for some } z.$$

CONVECTIVE FLOW

The Boussinesq approximation leads to

$$\nabla\cdot\boldsymbol{u}=0$$

$$\frac{D\boldsymbol{u}}{Dt}=-\frac{1}{\rho_0}\nabla p+(1-\alpha(T-T_0))\boldsymbol{g}+\upsilon\nabla^2\boldsymbol{u}$$

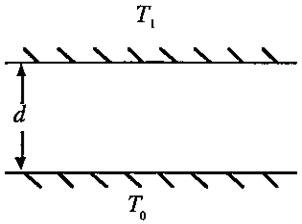
and

$$\frac{DT}{Dt}=\kappa\nabla^2 T$$

Rayleigh-Bénard convection

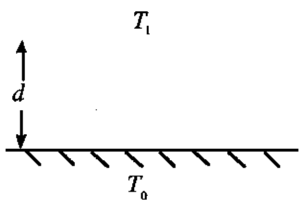
A fluid between two **rigid** plates is unstable when

$$\mathrm{Ra}\geq 1708$$



A liquid with a **free** upper surface is unstable when

$$\frac{\mathrm{Ra}}{\mathrm{Ra}_c}+\frac{\mathrm{Ma}}{\mathrm{Ma}_c}\geq 1$$



where

$$\mathrm{Ra}=\frac{g\alpha(T_0-T_1)d^3}{\upsilon\kappa},\quad \mathrm{Ma}=\frac{\chi(T_0-T_1)d}{\rho\upsilon\kappa}\quad \text{with } \chi=-\frac{d\sigma}{dT}$$

$\mathrm{Ra}_c\approx 670$ $\qquad$ $\mathrm{Ma}_c\approx 80.$

USEFUL MATHEMATICAL FORMULA

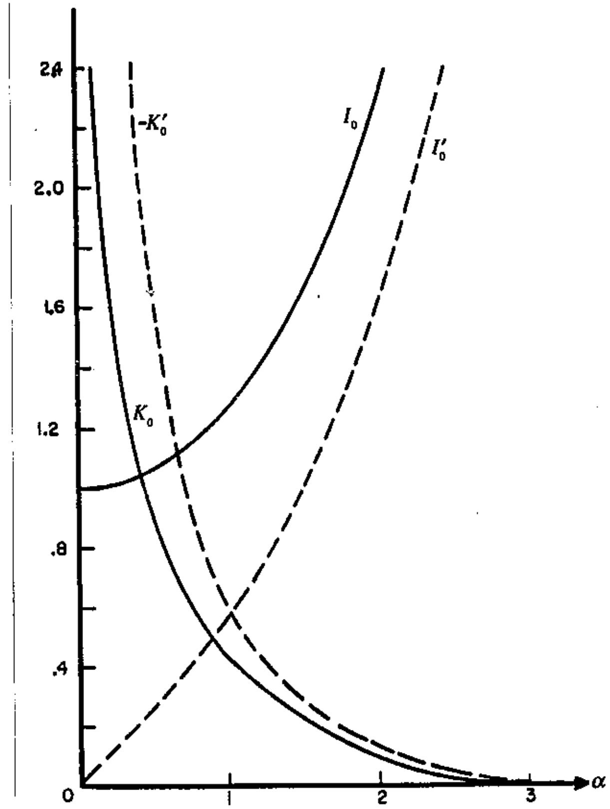
Modified Bessel equation

$I_0(kr)$ and $K_0(kr)$ are two independent solutions of

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-k^2f=0.$$

$I_0(kr)$ is finite at $r=0$ and tends to infinity as $r\to\infty$,

$K_0(kr)$ is infinite at $r=0$ and tends to zero as $r\to\infty$.



$I_0(\alpha)$, $K_0(\alpha)$, $I_0'(\alpha)$, $K_0'(\alpha)$
where $'$ denotes a derivative