Version EM/2

EGT3
ENGINEERING TRIPOS PART IIB

Friday 1 May 2015 2 to 3.30

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4A12 Data Card (3 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Show that for an inviscid fluid the helicity density, $\mathbf{u}\cdot\boldsymbol{\omega}=\mathbf{u}\cdot\nabla\times\mathbf{u}$, where $\mathbf{u}$ is the velocity, $\boldsymbol{\omega}$ is the vorticity, P is the pressure, and ρ is the density, is governed by

$$\frac{D}{Dt}(\mathbf{u}\cdot\boldsymbol{\omega})=\nabla\cdot\left[\left(\frac{\mathbf{u}^2}{2}-\frac{P}{\rho}\right)\boldsymbol{\omega}\right]$$

Hence confirm that the integral $H=\int\mathbf{u}\cdot\boldsymbol{\omega}dV$ over all space is conserved for a localised distribution of vorticity. [30%]

(b) A vorticity field, $\boldsymbol{\omega}(\mathbf{x},t)$, consists of two, thin, interlinked vortex tubes. They have vorticity fluxes Φ_1 and Φ_2 and centrelines C_1 and C_2.

(i) Confirm that the net helicity is given by

$$H=\Phi_1\oint_{C_1}\mathbf{u}\cdot(d\mathbf{l})+\Phi_2\oint_{C_2}\mathbf{u}\cdot(d\mathbf{l})$$

and use Stokes' theorem to show that, if the tubes are linked in a right-handed manner, $H=2\Phi_1\Phi_2$. [30%]

(ii) What is the value of H if the vortex tubes are not linked or are linked in a left-handed manner? [10%]

(c) State Helmholtz's two theorems of vortex dynamics and use these two theorems to explain why H is conserved in part (b). [20%]

(d) When the fluid is viscous, H in part (b) need not be conserved. Give a physical interpretation of this. [10%]

2 (a) In a steady, two-dimensional flow the temperature field $T(\mathbf{x})$ is governed by

$$\mathbf{u}\cdot\nabla T = \alpha\nabla^2 T$$

where α is the thermal diffusivity and $\mathbf{u}$ is the velocity.

(i) Show that, when $\alpha = 0$, $T(\mathbf{x})$ takes the form $T = T(\psi)$, where ψ is the streamfunction. Hence show that, when α is small but finite,

$$\mathbf{u}\cdot\nabla T = \alpha\nabla\cdot\left[T'(\psi)\nabla\psi\right]$$

where $T'(\psi) = dT/d\psi$. [20%]

(ii) Integrate this over the area *A* which is bounded by a closed streamline *C* and hence confirm that, for a small but finite α, $\alpha\, T'(\psi)\oint_C \nabla\psi\cdot\mathbf{n}dr = 0$, where $\mathbf{n}$ is the unit normal to the streamline and dr is part of the streamline. [20%]

(iii) Use the fact that $\nabla\psi$ is parallel to $\mathbf{n}$ at each point on the streamline to show that T is independent of position in the limit $\alpha \to 0$. [20%]

(b) (i) What is the physical interpretation of *T*=constant in Part (a-iii)? [10%]

(ii) It is often found that *T*=constant also holds when the flow is turbulent and steady (on average). Why is this? [10%]

(iii) State the Prandtl-Batchelor theorem and briefly discuss how the proof of *T*=constant in Part (a-iii) may be used to establish this theorem. [20%]

3 (a) Discuss the physical meaning and order of magnitude of the terms in the transport equation for the turbulent scalar fluctuations. Discuss the physical basis of the usual model for the mean scalar dissipation. [50%]

(b) In a situation where the mean scalar and its fluctuation energy are homogeneous, and where the turbulence is isotropic and homogeneous, estimate the time needed for the scalar energy to decay to 10% of its initial value in terms of eddy turnover times. [25%]

(c) Is the concept of statistically-steady homogeneous isotropic turbulence realistic? Give reasons for your answer. [25%]

4 Consider a flat plate boundary layer, where the plate is horizontal and hotter than the fluid flowing above it. In the governing equation for the turbulent kinetic energy (k) with buoyancy, the production term due to the fluctuating body force can be written as

$$\overline{g_i' u_i'} = \frac{g}{T} \overline{\theta' u_3'}$$

where T is the absolute temperature (taken as constant), g is the gravitational constant, θ' is the temperature fluctuation, and u_3' is the turbulent velocity fluctuation in the vertical direction.

(a) Neglecting turbulence production due to shear, which turbulent velocity component is generated due to buoyancy? What is the physical mechanism by which the other components receive energy? [30%]

(b) Assuming all terms in the k-equation are homogeneous in space and steady in time, show that the order of magnitude of the temperature fluctuations is $\frac{k}{L}\frac{T}{g}$, where L is the integral turbulent lengthscale. [40%]

(c) Now consider turbulence production due to shear, relax the assumption of spatial homogeneity, and assume that the mean temperature profile follows the mean streamwise velocity profile. Sketch qualitatively the expected distributions of the three normal Reynolds stresses and of the temperature fluctuations across the boundary layer and give reasons for your choices. [30%]

END OF PAPER

Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint \boldsymbol{A} \cdot d\boldsymbol{S}$

Stokes : $\int (\nabla \times A) \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

Vector Identities

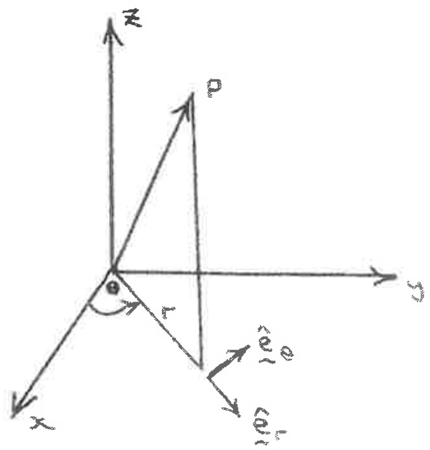
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

Cambridge University Engineering Department

4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned} \frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ & -\overline{u_i' u_j'} \frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'} \end{aligned}$$

The $k - \varepsilon$ model:

$$\frac{\partial k}{\partial t} + \overline{u}_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + P_k - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k} P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

$$P_k = \frac{1}{2}\nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)^2$$

$$C_\mu = 0.09,\ c_{\varepsilon 1} = 1.44,\ c_{\varepsilon 2} = 1.92,\ \sigma_k = 1.0,\ \sigma_\varepsilon = 1.3$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi'\phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D\frac{\partial^2 \sigma^2}{\partial x_j^2} - \overline{2\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'}\frac{\partial \overline{\phi}}{\partial x_j} - 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Scalar fluctuations (modelled):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_i \frac{\partial \sigma^2}{\partial x_i} = \frac{\partial}{\partial x_i}\left((D + D_{turb})\frac{\partial \sigma^2}{\partial x_i}\right) + 2D_{turb}\left(\frac{\partial \overline{\phi}}{\partial x_i}\right)^2 - 2\overline{N}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2 = 2\frac{u}{L_{turb}}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned}
\eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\
\tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\
v_K &= \left(\nu\varepsilon\right)^{1/4}
\end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned}
\overline{u_i' u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\
\overline{u_j' \phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j}
\end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1' u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$