EGT3
ENGINEERING TRIPOS PART IIB

Monday 18 April 2016   9.30 to 11

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4A12 Turbulence and Vortex Dynamics data sheet (3 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) What do we mean by *vortex stretching* and what is its role in determining the structure of the small (i.e. Kolmogorov) scales? [30%]

(b) Assume that the flow in a duct with a square cross section of side H can be taken as turbulent with homogeneous isotropic turbulence. At the exit of the duct, a converging nozzle is attached so that the nozzle's exit is a rectangle with sides H and H/C, where $C > 1$. Discuss how the eddy shape and the three components of the turbulent velocity fluctuations are altered at the exit of the nozzle relative to their characteristics at the entrance of the nozzle. Ignore wall friction effects. [40%]

(c) Consider again the square duct of Part (b) but with the nozzle removed. An infinite flat plate is placed normally to the flow at a distance $2H$ downstream of the exit. Sketch approximately the mean flow streamline pattern. Sketch and discuss the individual normal Reynolds stresses along the axis of the flow, focusing separately on the regions far from the plate (where wall friction effects are ignored) and close to it (where wall friction effects are not ignored). To facilitate the discussion and the sketching, assume that the region affected by the wall has a width of about $H/2$. [30%]

2 A chemical mixer is modelled as a closed volume with zero mean velocities in all directions and homogeneous isotropic turbulence with initial kinetic energy k_0. Mechanical power P_0 per unit fluid mass is injected to the mixer by the action of small fans. Neglect any mean flow created by the fans.

(a) Write down an estimate for the integral lengthscale L_0 when the fans are working. [20%]

(b) Is the assumption of homogeneous isotropic non-decaying turbulence everywhere in the vessel realistic? [30%]

(c) The power to the fans is now switched off. Assuming that the integral lengthscale does not change, derive an expression for the evolution of the turbulent kinetic energy k with time t. What is k/k_0 when t equals the initial eddy turnover time? [30%]

(d) Now assume that the lengthscale may change after the power to the fans is switched off. Using the $k-\varepsilon$ model as given in the Data Card and as applied to this flow, and taking $k \sim t^{-m}$ and $\varepsilon \sim t^{-n}$ with $m = 3/2$, find n. [20%]

3 (a) Explain, with the aid of sketches, why there is a radial component of motion in a Karman layer and in a Bodewadt layer. Why is there an axial velocity external to the boundary layer in both cases? [30%]

(b) Through a consideration of simple force balances, show that in a Karman layer

$$\frac{u_\theta^2}{r} \sim \nu \frac{u_r}{\delta^2}$$

where δ is the boundary layer thickness, ν the kinematic viscosity, and we have used cylindrical polar coordinates (r,θ). Hence obtain an order of magnitude estimate for the boundary layer thickness in terms of ν and the rate of rotation of the boundary, Ω. Now show that the axial flow external to the boundary layer scales as $| u_z | \sim \sqrt{\nu\Omega}$. [35%]

(c) Torque is transmitted between two large, parallel disks that rotate at different speeds about a common vertical axis. The gap between the disks is filled with viscous oil. The upper disk rotates at Ω_0, the lower disk is stationary, and the gap between the disks is much greater than the boundary layer thickness on the disks. Sketch the secondary flow pattern between the disks. Outside the boundary layers, the fluid between the disks rotates at a rate somewhat less than Ω_0; find that rotation rate. *Hint: you may use the fact that, outside a Karman layer, the axial velocity is* $| u_z | = 0.9\sqrt{\nu\Omega}$, *where* Ω *is the rotation rate of the surface relative to the remote fluid, while that outside a Bodewadt layer is* $| u_z | = 1.4\sqrt{\nu\Omega}$, *where* Ω *is the rotation rate relative to the stationary surface.* [35%]

4 (a) A two-dimensional incompressible flow is generated by two line vortices, which are free to move. The velocity of the flow tends to zero at large distances from the vortices, whose strengths are Γ_1 and Γ_2. Show that the vortices rotate as a pair at a constant separation distance d and about a fixed point on the straight line between them. (Note that the velocity of the fluid at this fixed point need not be zero.) In terms of Γ_1, Γ_2, and d, find the position of the centre of rotation relative to the vortices and determine the rate of rotation of the pair. [40%]

(b) What is the motion of the pair when $\Gamma_1 = -\Gamma_2$? [10%]

(c) Consider a vortex pair with $\Gamma_1 = \Gamma_2$ in the semi-infinite plane $y \geq 0$. At a particular instant, the line joining the vortices is parallel to the boundary (the x-axis), and is at a distance D from that boundary, with $D > d$. Give a qualitative description of the motion of the vortex pair. [30%]

(d) Describe an experiment to illustrate the interaction between two coaxial vortex rings, which is analogous to the behaviour considered in Part (c). [20%]

END OF PAPER

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Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint A \cdot dS$

Stokes : $\int (\nabla \times A) \cdot d\boldsymbol{S} = \oint A \cdot d\boldsymbol{l}$

Vector Identities

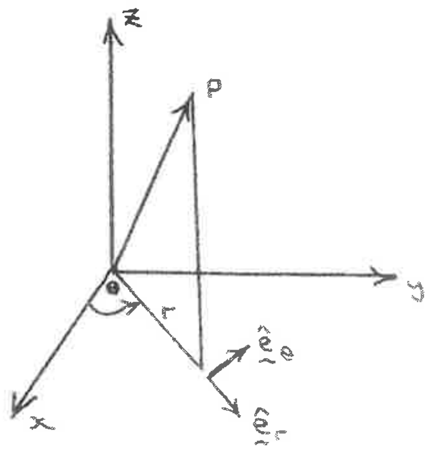
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

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4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned} \frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ & -\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'} \end{aligned}$$

The $k - \varepsilon$ model:

$$\frac{\partial k}{\partial t} + \overline{u}_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + P_k - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k}P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

$$P_k = \frac{1}{2}\nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)^2$$

$$C_\mu = 0.09,\ c_{\varepsilon 1} = 1.44,\ c_{\varepsilon 2} = 1.92,\ \sigma_k = 1.0,\ \sigma_\varepsilon = 1.3$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi'\phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D\frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'}\frac{\partial \overline{\phi}}{\partial x_j} - 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Scalar fluctuations (modelled):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_i \frac{\partial \sigma^2}{\partial x_i} = \frac{\partial}{\partial x_i}\left((D + D_{turb})\frac{\partial \sigma^2}{\partial x_i}\right) + 2D_{turb}\left(\frac{\partial \overline{\phi}}{\partial x_i}\right)^2 - 2\overline{N}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2 = 2\frac{u}{L_{turb}}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned}
\eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\
\tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\
v_K &= \left(\nu\varepsilon\right)^{1/4}
\end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned}
\overline{u_i' u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\
\overline{u_j' \phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j}
\end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1' u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$