Version pad/3

EGT3
ENGINEERING TRIPOS PART IIB

Monday 23 April 2018 2.00 to 3.40

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book
Attachment: 4A12 Turbulence and Vortex Dynamics Data Card (3 pages).

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 The air motion in an auditorium of dimensions 30 m $\times$ 30 m in plan and 10 m high is to be maintained by the action of small fans in a state of homogeneous isotropic turbulence with integral lengthscale 0.5 m and characteristic turbulent velocity fluctuation 0.5 m/s. Assume atmospheric conditions and neglect buoyancy.

(a) Estimate the power input to maintain this state. [30%]

(b) A small fire erupts on the floor in the centre of the auditorium, and emits smoke for a short period of time before being extinguished. Estimate the order of magnitude of the time taken until a smoke detector at the ceiling immediately above the fire will first detect the fire. [30%]

(c) The small fans are now switched off and hence there is no power input to the air in the auditorium. If the integral lengthscale stays constant, estimate the time needed for the turbulence intensity to fall to 10% of its initial value.

[40%]

2 Consider a turbulent, round, axisymmetric jet exiting a nozzle of diameter D with mean velocity U.

(a) What is meant by the term *self-similar*? [25%]

(b) The characteristic lengthscale of the jet grows as x^1, where x is the streamwise distance from the nozzle. Show that the mean centreline velocity decays as x^{-1}. [25%]

(c) If the jet fluid carries a passive scalar c, find how the mean value of c at the centreline scales with x. Sketch the mean streamlines of the flow and typical radial profiles of the mean velocity components and the turbulent intensity and hence explain the mixing processes in the jet.

[50%]

3 Consider the vortex sheet

$$\boldsymbol{\omega} = \frac{\Phi}{\sqrt{\pi}\ell(t)} \exp\left[-\left(x/\ell(t)\right)^2\right]\hat{\mathbf{e}}_z \qquad (1)$$

which corresponds to the unidirectional flow $\mathbf{u}^{(\mathrm{vort})} = u_y(x,t)\hat{\mathbf{e}}_y$. Here $\ell(t)$ is the characteristic thickness of the vortex sheet, Φ is a constant, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ are unit vectors in (x, y, z) Cartesian coordinates. This vortex sheet sits in the straining flow $\mathbf{u}^{(\mathrm{strain})} = (-\alpha x, 0, \alpha z)$, where α is a constant.

(a) Sketch the vortex sheet and the straining flow and confirm that $\mathbf{u}^{(\mathrm{strain})}$ is both solenoidal and irrotational.

[10%]

(b) Confirm that Φ is the flux of vorticity per unit length of the sheet and explain why it is constant even when the flow is unsteady. You may need to make use of the definite integral

$$\int_{-\infty}^{\infty} \exp(-s^2)ds = \sqrt{\pi}$$ [15%]

(c) Show that a steady solution of the vorticity equation,

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}$$

of the form (1) is possible for an appropriate choice of ℓ, say ℓ_0, and find the relationship between ℓ_0, α and ν. [40%]

(d) Outline what physical processes must balance in the steady solution of part (c). [15%]

(e) Discuss what happens to $\ell(t)$ if, for a given α and ν, it initially exceeds ℓ_0, and what happened if it is initially smaller than ℓ_0. [20%]

4 (a) Consider the Bödewadt layer on an infinite, plane surface. Use dimensional analysis to show that the boundary layer thickness, δ, must scale as $\delta \sim \sqrt{\nu/\Omega}$, where ν is the kinematic viscosity and Ω the rotation rate of the fluid outside the boundary layer.

[15%]

(b) Sketch the secondary flow in the Bödewadt layer and explain why there is a radial flow inside the boundary layer and an axial flow external to the layer.

[20%]

(c) Given that the radial and azimuthal velocity components in the boundary layer are of similar magnitudes, $u_r \sim u_\theta \sim \Omega r$, show that the axial velocity just outside the boundary layer is independent of radius and of magnitude $u_z \sim \sqrt{\nu\Omega}$.

[20%]

(d) A stationary cylindrical tank of radius R is partially filled with water. The water is set into rotation with angular velocity Ω_0 and then allowed to slowly spin down through the action of friction on the walls of the tank. The rotation rate of the water is sufficiently slow for the flow to remain laminar and it is required to determine the core flow $\mathbf{u}^{(\text{core})}$ outside the Bödewadt and side-wall boundary layers during spin down. Polar coordinates (r, θ, z) are centred on the tank with $z = 0$ marking the top of the Bödewadt layer and $z = H$ the free surface. The motion is axisymmetric, the depth and tank radius are similar, $H \sim R$, and the core flow takes the form $\left(u_r^{(\text{core})}, \Omega(t)r, u_z^{(\text{core})}\right)$.

(i) Use continuity of mass to show that

$$u_r^{(\text{core})} H \sim (\Omega r)\delta$$

where δ is the Bödewadt boundary layer thickness.

(ii) Use the above scaling laws as well as continuity to show that

$$u_z^{(\text{core})} \sim \sqrt{\nu\Omega}\left[1 - z/H\right]$$

(iii) Estimate the time taken for the fluid to stop spinning in terms of the parameters ν, Ω and H.

[45%]

END OF PAPER

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Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint \boldsymbol{A} \cdot d\boldsymbol{S}$

Stokes : $\int (\nabla \times A) \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

Vector Identities

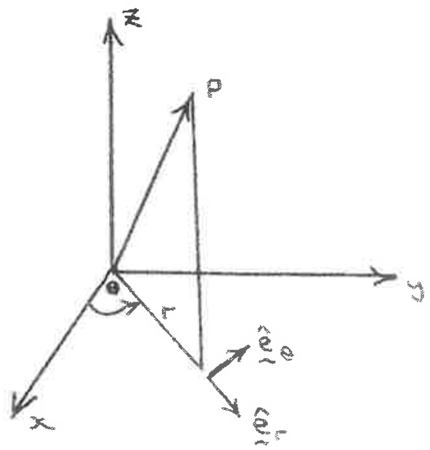
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (f\boldsymbol{A}) = f(\nabla \cdot \boldsymbol{A}) + \boldsymbol{A} \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \right)$$

Cambridge University Engineering Department

4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ &\quad - \overline{u_i' u_j'} \frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}\end{aligned}$$

The $k - \varepsilon$ model:

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_i \frac{\partial k}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + P_k - \varepsilon \\ \frac{\partial \varepsilon}{\partial t} + \overline{u}_i \frac{\partial \varepsilon}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k} P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}\end{aligned}$$

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

$$P_k = \frac{1}{2}\nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)^2$$

$$C_\mu = 0.09,\ c_{\varepsilon 1} = 1.44,\ c_{\varepsilon 2} = 1.92,\ \sigma_k = 1.0,\ \sigma_\varepsilon = 1.3$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi'\phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D\frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'}\frac{\partial \overline{\phi}}{\partial x_j} - 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Scalar fluctuations (modelled):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_i \frac{\partial \sigma^2}{\partial x_i} = \frac{\partial}{\partial x_i}\left((D + D_{turb})\frac{\partial \sigma^2}{\partial x_i}\right) + 2D_{turb}\left(\frac{\partial \overline{\phi}}{\partial x_i}\right)^2 - 2\overline{N}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2 = 2\frac{u}{L_{turb}}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned} \eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\ \tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\ v_K &= \left(\nu\varepsilon\right)^{1/4} \end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned} \overline{u_i' u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\ \overline{u_j'\phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j} \end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1' u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$