EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 29 April 2015 2 to 3:30

Module 4C2

DESIGNING WITH COMPOSITES

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: Module 4C2 Designing with Composites data sheet (6 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Describe briefly what is meant by a laminate and explain why laminates are commonly employed in preference to a unidirectional lamina. What is a symmetric laminate and how does this symmetry affect the laminate's elastic response? [10%]

(b) Consider a Kevlar-epoxy unidirectional lamina with the elastic constants $E_1 = 76$ GPa, $E_2 = 5.5$ GPa, $G_{12} = 2.3$ GPa and $\nu_{12} = 0.34$, where the fibres are aligned with the 1 direction.

(i) Find the values of the loading angle θ relative to the fibre axis for which the compliance $\bar{S}_{16}$ of the lamina equals zero. Explain the importance of the $\bar{S}_{16}$ term in the lamina's elastic response. [25%]

(ii) Calculate the shear strain induced in the lamina when a tensile stress of 200 MPa is applied at an angle of 60° to the fibre axis. [15%]

(c) A $[\pm 45]_S$ laminate is made from 4 laminae, each of thickness t. Each lamina has the following elastic constants: $E_2 = 0.1E_1$, $G_{12} = 0.05E_1$ and $\nu_{12} = 0.3$. Find expressions for the laminate extensional stiffness matrix $[A]$ and the laminate coupling stiffness matrix $[B]$ in terms of E_1 and t. [50%]

2 Consider the design of a tubular drive shaft for a cooling tower. The tube has a radius of 50 mm and the tube wall is a 4 mm thick balanced symmetric laminate made from ±45° prepreg plies of AS/3501 CFRP composite (properties on the datasheet).

(a) Why might CFRP be a good choice of material for this application? [10%]

(b) Estimate the torque that the drive shaft can transmit without failing:

(i) using the data in Table 1 on the datasheet; [25%]

(ii) using laminate plate theory and a Tsai-Hill failure criterion. [55%]

(c) Comment on any differences observed in your predictions for the failure torque in parts (i) and (ii) of part (b). [10%]

3 (a) Describe methods used to measure resistance to delamination of fibre composites. [15%]

(b) Figure 1 illustrates a sharp notch of length 50 mm running perpendicular to the 0° direction in a $[0/90]_S$ cross-ply laminate made from four 0.25 mm thick plies of Scotchply/1002 GFRP (properties on the datasheet). The laminate is subject to a direct stress σ and a shear stress τ parallel and perpendicular to the 0° direction, respectively. Show that the laminate compliance matrix is given by

$$\begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} = \begin{bmatrix} 0.042 & -0.0038 & 0 \\ -0.0038 & 0.042 & 0 \\ 0 & 0 & 0.24 \end{bmatrix} \text{GPa}^{-1}$$ [30%]

(c) The laminate has toughnesses $G_{IC} = 20$ kJ/m^2 and $G_{IIC} = 40$ kJ/m^2 for the crack geometry of Fig. 1. Use linear elastic fracture mechanics (LEFM), stating any further assumptions you need to make, to predict the applied stresses giving failure at the notch associated with:

(i) a direct stress σ applied alone;

(ii) a shear stress τ applied alone;

(iii) a direct stress σ and a shear stress $\tau = 0.5\sigma$ applied simultaneously. [35%]

(d) Discuss the applicability of LEFM to fracture in fibre composites. [20%]

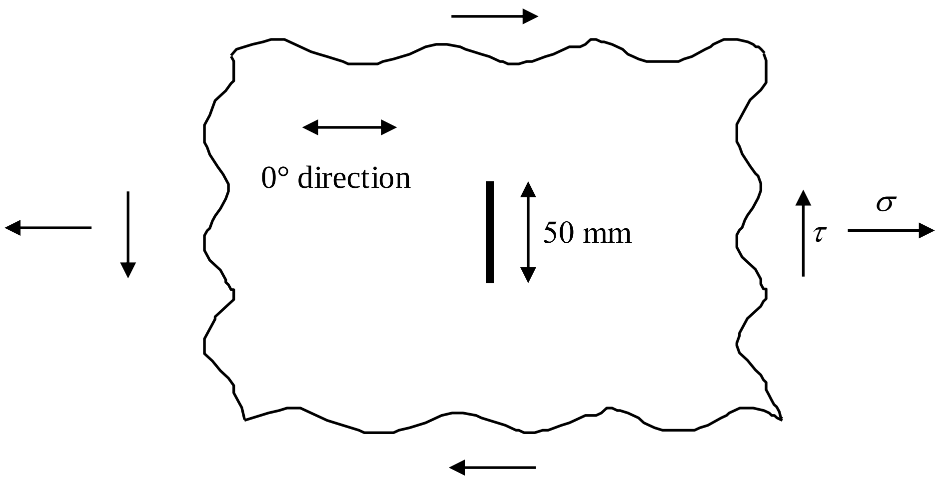


Fig. 1

4 (a) Discuss the design and manufacture of a long fibre composite handle used, in conjunction with a brush on the end of the handle, for cleaning the outside windows on upper floors of buildings. Include the following issues in your discussion (but do not limit yourself to just these points):

– material and layup;

– manufacturing route, including a detailed description of your suggested method;

– structural design and any feature details. [55%]

(b) Why do woven composites tend to have lower stiffness and compressive strength than an equivalent composite made of prepreg with the same fibres and matrix? [15%]

(c) Why is barely visible impact damage an important issue in aerospace design? [15%]

(d) Why are septic tanks (underground tanks used as part of a small-scale sewage treatment system) commonly made of GFRP? [15%]

END OF PAPER

ENGINEERING TRIPOS PART II B

Module 4C2 – Designing with Composites

DATA SHEET

The in-plane compliance matrix [S] for a transversely isotropic lamina is defined by

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = [S] \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} \qquad \text{where } [S] = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix}$$

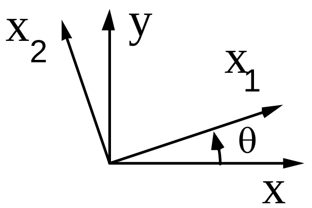
[S] is symmetric, giving $\nu_{12}/E_1 = \nu_{21}/E_2$. The compliance relation can be inverted to give

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix}$$

where
$$\begin{aligned} Q_{11} &= E_1/(1-\nu_{12}\nu_{21}) \\ Q_{22} &= E_2/(1-\nu_{12}\nu_{21}) \\ Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21}) \\ Q_{66} &= G_{12} \end{aligned}$$

Rotation of co-ordinates

Assume the principal material directions (x_1, x_2) are rotated anti-clockwise by an angle θ , with respect to the (x, y) axes.



Then,
$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix} = [T] \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = [T]^{-T} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

where
$$[\mathrm{T}] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

and
$$[\mathrm{T}]^{-T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & (\cos^2\theta - \sin^2\theta) \end{bmatrix}$$

The stiffness matrix $[Q]$ transforms in a related manner to the matrix $[\overline{Q}]$ when the axes are rotated from (x_1, x_2) to (x, y)

$$[\overline{Q}] = [T]^{-1}[Q][T]^{-T}$$

In component form,

$$[\overline{Q}] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \quad \text{where} \quad \begin{aligned} \overline{Q}_{11} &= Q_{11}c^4 + Q_{22}s^4 + 2\left(Q_{12} + 2Q_{66}\right)s^2c^2 \\ \overline{Q}_{12} &= \left(Q_{11} + Q_{22} - 4Q_{66}\right)s^2c^2 + Q_{12}\left(c^4 + s^4\right) \\ \overline{Q}_{22} &= Q_{11}s^4 + Q_{22}c^4 + 2\left(Q_{12} + 2Q_{66}\right)s^2c^2 \\ \overline{Q}_{16} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)sc^3 - \left(Q_{22} - Q_{12} - 2Q_{66}\right)s^3c \\ \overline{Q}_{26} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)s^3c - \left(Q_{22} - Q_{12} - 2Q_{66}\right)sc^3 \\ \overline{Q}_{66} &= \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)s^2c^2 + Q_{66}\left(s^4 + c^4\right) \\ &\text{with } c = \cos\theta, \quad s = \sin\theta \end{aligned}$$

The compliance matrix $[S] \equiv [Q]^{-1}$ transforms to $[\overline{S}] \equiv [\overline{Q}]^{-1}$ under a rotation of co-ordinates by θ from (x_1, x_2) to (x, y), as

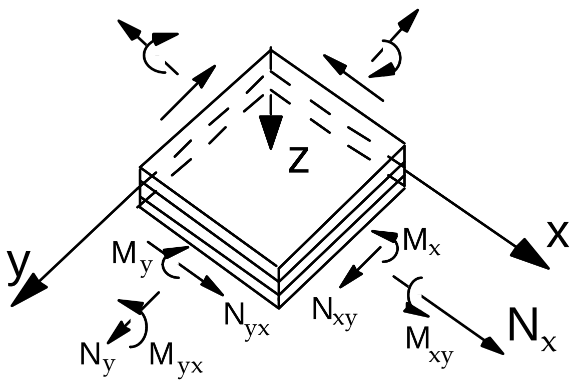
$$[\overline{S}] = [T]^T [S][T]$$

and in component form,

$$\begin{aligned} \overline{S}_{11} &= S_{11}c^4 + S_{22}s^4 + \left(2S_{12} + S_{66}\right)s^2c^2 \\ \overline{S}_{12} &= S_{12}\left(c^4 + s^4\right) + \left(S_{11} + S_{22} - S_{66}\right)s^2c^2 \\ \overline{S}_{22} &= S_{11}s^4 + S_{22}c^4 + \left(2S_{12} + S_{66}\right)s^2c^2 \\ \overline{S}_{16} &= \left(2S_{11} - 2S_{12} - S_{66}\right)sc^3 - \left(2S_{22} - 2S_{12} - S_{66}\right)s^3c \\ \overline{S}_{26} &= \left(2S_{11} - 2S_{12} - S_{66}\right)s^3c - \left(2S_{22} - 2S_{12} - S_{66}\right)sc^3 \\ \overline{S}_{66} &= \left(4S_{11} + 4S_{22} - 8S_{12} - 2S_{66}\right)s^2c^2 + S_{66}\left(c^4 + s^4\right) \end{aligned}$$

with $c = \cos\theta, \quad s = \sin\theta$

Laminate Plate Theory



Consider a plate subjected to stretching of the mid-plane by $\left(\varepsilon_x^o, \varepsilon_y^o, \varepsilon_{xy}^o\right)^T$ and to a curvature $\left(\kappa_x, \kappa_y, \kappa_{xy}\right)^T$. The stress resultants $\left(N_x, N_y, N_{xy}\right)^T$ and bending moment per unit length $\left(M_x, M_y, M_{xy}\right)^T$ are given by

$$\begin{pmatrix} N \\ \cdots \\ M \end{pmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & . & \cdots \\ B & \vdots & D \end{bmatrix} \begin{pmatrix} \varepsilon^o \\ \cdots \\ \kappa \end{pmatrix}$$

In component form, we have,

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

where the laminate extensional stiffness, A_{ij}, is given by:

$$A_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k dz = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k - z_{k-1}\right)$$

the laminate coupling stiffnesses is given by

$$B_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z dz = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right)$$

and the laminate bending stiffness are given by:

$$D_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z^2 dz = \frac{1}{3}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right)$$

with the subscripts i, j = 1, 2 or 6.

Here,

n = number of laminae

t = laminate thickness

$z_{k\text{-}1}$ = distance from middle surface to the top surface of the *k-th* lamina

z_k = distance from middle surface to the bottom surface of the *k-th* lamina

Quadratic failure criteria.

For plane stress with $\sigma_3 = 0$, failure is predicted when

Tsai-Hill: $\frac{\sigma_1^2}{s_L^2} - \frac{\sigma_1\sigma_2}{s_L^2} + \frac{\sigma_2^2}{s_T^2} + \frac{\tau_{12}^2}{s_{LT}^2} \geq 1$

Tsai-Wu: $F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 \geq 1$

where $F_{11} = \frac{1}{s_L^+ s_L^-}$, $F_{22} = \frac{1}{s_T^+ s_T^-}$, $F_1 = \frac{1}{s_L^+} - \frac{1}{s_L^-}$, $F_2 = \frac{1}{s_T^+} - \frac{1}{s_T^-}$, $F_{66} = \frac{1}{s_{LT}^2}$

F_{12} should ideally be optimised using appropriate strength data. In the absence of such data, a default value which should be used is

$$F_{12} = -\frac{\left(F_{11}F_{22}\right)^{1/2}}{2}$$

Fracture mechanics

Consider an orthotropic solid with principal material directions x_1 and x_2. Define two effective elastic moduli E'_A and E'_B as

$$\frac{1}{E'_A} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2} \left(\left(\frac{S_{22}}{S_{11}}\right)^{1/2} \left(1 + \frac{2S_{12} + S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

$$\frac{1}{E'_B} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2} \left(\left(\frac{S_{11}}{S_{22}}\right)^{1/2} \left(1 + \frac{2S_{12} + S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

where S_{11} etc. are the compliances.

Then G and K are related for plane stress conditions by:

crack running in x_1 direction: $G_I E'_A = K_I^2 ; G_{II} E'_B = K_{II}^2$

crack running in x_2 direction: $G_I E'_B = K_I^2 ; G_{II} E'_A = K_{II}^2$.

For mixed mode problems, the total strain energy release rate G is given by

$G = G_I + G_{II}$

Approximate design data

	Steel	Aluminium	CFRP	GFRP	Kevlar
Cost C (£/kg)	1	2	100	5	25
E_1 (GPa)	210	70	140	45	80
G (GPa)	80	26	≈35	≈11	≈20
ρ (kg/m^3)	7800	2700	1500	1900	1400
e^+ (%)	0.1-0.8	0.1-0.8	0.4	0.3	0.5
e^- (%)	0.1-0.8	0.1-0.8	0.5	0.7	0.1
e_{LT} (%)	0.15-1	0.15-1	0.5	0.5	0.3

Table 1. Material data for preliminary or conceptual design. Costs are very approximate.

	Aluminium	Carbon/epoxy (AS/3501)	Kevlar/epoxy (Kevlar 49/934)	E-glass/epoxy (Scotchply/1002)
Cost (£/kg)	2	100	25	5
Density (kg/m^3)	2700	1500	1400	1900
E_1 (GPa)	70	138	76	39
E_2 (GPa)	70	9.0	5.5	8.3
ν_{12}	0.33	0.3	0.34	0.26
G_{12} (GPa)	26	6.9	2.3	4.1
s_L^+ (MPa)	300 (yield)	1448	1379	1103
s_L^- (MPa)	300	1172	276	621
s_T^+ (MPa)	300	48.3	27.6	27.6
s_T^- (MPa)	300	248	64.8	138
s_{LT} (MPa)	300	62.1	60.0	82.7

Table 2. Material data for detailed design calculations. Costs are very approximate.

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Engineering Tripos Part IIB: Module 4C2
Designing with Composites

Numerical answers - 2014/15

1. (b) (i) 0, 68, 90°, (ii) ε_{xy}= 0.0046 or ε_{12}= 0.038

(c) $[A] = \begin{bmatrix} 1.37 & 0.97 & 0 \\ 0.97 & 1.37 & 0 \\ 0 & 0 & 1.05 \end{bmatrix} tE_1$, $[B] = 0$

2. (b) (i) 11.0 kN m, (ii) 25.4 kN m

3. (c) (i) 66 MPa, (ii) 94 MPa, (iii) 73 MPa, assuming that G_C=(G_{IC} + G_{IIC})/2.