Version RSL/4

EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 20 April 2016 14.00 to 15.30

Module 4C7

RANDOM AND NON-LINEAR VIBRATIONS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C7 Random and Non-linear Vibrations data sheet (4 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Explain the meaning of the terms “stationary” and “ergodic” in the context of a random process. Explain whether a stationary process can be non-ergodic, and conversely whether an ergodic process can be non-stationary. [20%]

(b) The envelope $a(t)$ of a random process $x(t)$ is a smooth curve that joins the peaks of $x(t)$. The envelope is always positive, $a(t) > 0$. The joint probability density function of the envelope and its time derivative is given by

$$p(a, \dot{a}) = Ca \exp\left[-a^2 / (2\alpha^2) - \dot{a}^2 / (2\beta^2)\right],$$

where C, α and β are constants.

(i) By applying the normalisation condition, find an expression for the constant C in terms of the constants α and β. [15%]

(ii) Find an expression for the probability density function of $a(t)$. [15%]

(iii) Find an expression for the probability that the envelope is less than a specified value b at a specified time t_0. [15%]

(iv) By considering the level crossing rates of the envelope, find an expression for the probability that the envelope is less than a specified value b throughout a time interval of duration T. [35%]

2 The wind loading $F(t)$ acting on a wind turbine blade is a random process that is governed by the equation

$$\dot{F} + \alpha F = \alpha w(t),$$

where α is a constant and $w(t)$ is a Gaussian white noise random process with double-sided spectral density S_0. The equation of motion that governs the amplitude $x(t)$ of the first vibration mode of a turbine blade is given by

$$\ddot{x} + 2\beta\omega_n\dot{x} + \omega_n^2 x = \gamma F(t),$$

where ω_n is the undamped natural frequency, β is the damping ratio, and γ is a constant.

(a) Derive an expression for the spectrum of the wind force and for the spectrum of the blade response. [25%]

(b) Derive expressions for the root-mean-square values of $x(t)$ and $\dot{x}(t)$ for the case where $\alpha \square \omega_n$. What is the root-mean-square value of the force in this case? [25%]

(c) Fatigue failure at the root of the blade is governed by the S-N law

$$N(S) = \gamma / S,$$

where $N(S)$ is the number of cycles to failure under stress cycles of amplitude S, and γ is a constant. A peak of amplitude b in the response $x(t)$ gives rise to a stress peak of amplitude $S = \lambda b$. For the case $\alpha \square \omega_n$ derive an expression for the average fatigue damage per unit time caused by the wind loading. [30%]

(d) List the assumptions that are involved in your answer to part (c), and suggest an appropriate factor of safety. [20%]

3 The equation of motion of a single-degree-of-freedom nonlinear dynamic system has the form

$$\ddot{x} - a_1 x^3 + a_2 x^5 = 0,$$

where a_1 and a_2 are constants, with $a_2 > 0$.

(a) Show that the equation of motion could approximately represent the frictionless sliding of a bead along a wire under the influence of gravity. Sketch the shape of the wire for each of the two cases $a_1 > 0$ and $a_1 < 0$. [20%]

(b) For each of the two cases $a_1 > 0$ and $a_1 < 0$ determine the singular points for the system, and determine the type and stability of each singular point. [40%]

(c) Sketch the behaviour of the system in the phase plane for each of the two cases $a_1 > 0$ and $a_1 < 0$. [20%]

(d) Relate your answers to parts (a) and (c) by showing how various trajectories in the phase plane can be obtained by starting the bead with zero velocity at appropriate points on the wire. [20%]

4 An oscillator with nonlinear stiffness and linear damping has the equation of motion

$$\ddot{x} + b\dot{x} + cx|x| = B\cos\omega t \text{ ,}$$

where *B, b, c,* and ω are constants.

(a) By using the Describing Function approach, linearise the equation of motion on the assumption that the response amplitude has a value *A*. [40%]

(b) Show that the period of *undamped free* vibration $(b = B = 0)$ is inversely proportional to the square root of the vibration amplitude. [10%]

(c) Derive an expression for the amplitude of *undamped forced* vibration $(b = 0,\ \ B > 0)$ and show that there are three possible solutions for the amplitude when $\omega^4 > 32Bc/(3\pi)$. Explain this result in physical terms. [30%]

(d) Derive an equation that governs the amplitude of the *damped forced* vibration. [20%]

END OF PAPER

Answers

1. b) (i) $C = (\alpha^2 \beta \sqrt{2\pi})^{-1}$, (ii) $p(a) = (a / \alpha^2) e^{-0.5(a/\alpha)^2}$, (iii) $P = 1 - e^{-0.5(b/\alpha)^2}$
(iv) $P = \exp\left\{-T\beta b e^{-0.5(b/\alpha)^2} / (\alpha^2 \sqrt{2\pi})\right\}$

2. a) $S_{FF}(\omega) = \dfrac{\alpha^2 S_0}{\omega^2 + \alpha^2}$, $S_{xx}(\omega) = \dfrac{\alpha^2 \gamma^2 S_0}{\left[(\omega_n^2 - \omega^2)^2 + (2\beta\omega\omega_n)^2\right](\omega^2 + \alpha^2)}$
b) $\sigma_x^2 = \dfrac{\pi\gamma^2 S_0}{2\beta\omega_n^3}$, $\sigma_{\dot{x}}^2 = \dfrac{\pi\gamma^2 S_0}{2\beta\omega_n}$
c) $\mathrm{E}[D] = \lambda\omega_n\sigma_x / (2\gamma\sqrt{2\pi})$

3. b) For $a_1 < 0$: singular point is (0,0), a centre.
For $a_1 > 0$: (0,0), is a saddle point, and $(\pm\sqrt{a_1 / a_2}, 0)$ are centres.

4. a) Linearised equation is $\ddot{x} + b\dot{x} + [8c|A| / (3\pi)]x = B\cos\omega t$
d) $\left\{-\omega^2 + bi\omega + 8c|A| / (3\pi)\right\} A = B$, with A complex.