EGT3
ENGINEERING TRIPOS PART IIB

Monday 25 April 2016   2 to 3.30

Module 4C9

CONTINUUM MECHANICS

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C9 datasheet (3 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Using indicial notation, prove the following identities, where $\underline{a}$, $\underline{b}$ and $\underline{c}$ are vectors, each a function of position $\underline{x}$:

(i) if tensor $\underline{a} \otimes \underline{b}$ is symmetric, $(\underline{a} \otimes \underline{b}) \cdot \underline{c} = \underline{c} \cdot (\underline{a} \otimes \underline{b})$; [10%]

(ii) $\nabla \times (\nabla \times \underline{a}) = \nabla (\nabla \cdot \underline{a}) - (\nabla \cdot \nabla) \underline{a}$; [20%]

(iii) $\oint_S (\underline{a} \otimes \underline{b}) \cdot \underline{n} \, dS = \int_V [\, \underline{b} \cdot \nabla \underline{a} + \underline{a} (\nabla \cdot \underline{b}) \,] \, dV$,

where S is a closed surface with unit normal $\underline{n}$ enclosing a volume V . [20%]

(b) Figure 1 shows the steady-state, plane strain extrusion of a ductile metal, to reduce its thickness from $4h$ to $2h$. Assume that the material has a shear yield strength k, and that friction is negligible.

(i) Use the upper-bound method, with planes of tangential velocity discontinuity as shown by dashed lines in Fig. 1, to calculate the dependence of the extrusion force F per unit depth upon the variable x. Hence obtain the best upper bound. [40%]

(ii) Why is the above solution not exact? [10%]

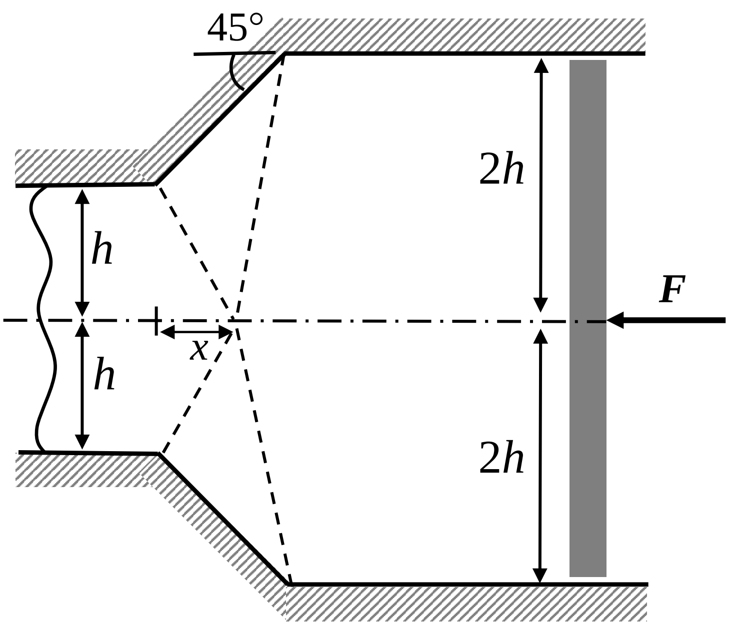

Fig. 1

2 (a) The uniaxial tensile response of a linear viscoelastic material satisfies the differential equation: $\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t)$, where E and η are material parameters, ε is the strain, $\dot{\varepsilon}$ the strain rate, and t is time.

(i) Show that the creep compliance $J_c(t) = \varepsilon(t)/\sigma_0$ (where σ_0 is a constant applied stress) is of the form:

$$J_c(t) = A\left(1 - e^{Bt}\right) ,$$

and derive expressions for the constants A and B. [20%]

(ii) Assuming the material to be initially stress free, and the Poisson's ratio ν to be time independent, the strain components $\varepsilon_{ij}(t)$ resulting from an arbitrary stress history $\sigma_{ij}(t)$ are given by the 3D convolution principle:

$$\varepsilon_{ij}(t) = (1+\nu)\int_0^t J_c(t-\tau)\frac{\partial\sigma_{ij}(\tau)}{\partial\tau}\,d\tau - \nu\int_0^t J_c(t-\tau)\frac{\partial\sigma_{kk}(\tau)}{\partial\tau}\delta_{ij}\,d\tau ,$$

where δ_{ij} is the usual Kronecker delta. A thin-walled sphere (radius R, wall thickness h) of the viscoelastic material is subjected to an internal pressure $p(t) = 0$ for $t < 0$ and $p(t) = \dot{p}t$ for $t \geq 0$, where $\dot{p}$ is a constant. Derive an expression for the rate of change of the radius, $\partial R/\partial t$. Sketch your solution, and mark on it the equivalent result for a linear elastic material. [40%]

(b) (i) Explain what Drucker meant by the stability of an elastic-plastic solid. What are the implications of stability upon the shape of the yield surface, and upon the direction of plastic flow with regard to the yield surface? [20%]

(ii) Explain why the yield surface of a fully dense metal is insensitive to the applied pressure, whereas the yield surface of a porous metal is pressure-dependent. [20%]

3 (a) A thin-walled tube has radius R, length L and wall thickness h. The material is isotropic and linear elastic, such that the strain components ε_{ij} are related to the stress components σ_{ij} by:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \,,$$

where E is the Young's modulus, ν is the Poisson's ratio and δ_{ij} is the usual Kronecker delta. The tube is subjected to an axial tensile force F and a torque T, and can be assumed to be in plane stress.

(i) Derive an expression for the elastic strain energy of the tube as a function of the axial strain ε_{11} and the engineering shear strain $\gamma = 2\varepsilon_{12}$. [15%]

(ii) Given that the ends of the tube undergo a relative rotation $\theta = \gamma L/R$ and a relative axial displacement $\Delta = \varepsilon_{11}L$, use the method of minimum potential energy to derive equations relating ε_{11} and γ to the force F and torque T, the tube geometry and material parameters. [20%]

(b) A thin-walled tube, with the same dimensions as in part (a), is now made from a metal with Young's modulus E, an initial uniaxial yield strength σ_Y, and a constant post-yield tangent modulus E_T. It yields in accordance with J_2 flow theory, with isotropic hardening, such that the uniaxial stress σ versus strain ε relation obeys

$$\varepsilon = \frac{\sigma}{E} \qquad \text{for } 0 \le \sigma \le \sigma_Y$$

$$\varepsilon = \frac{\sigma_Y}{E} + \frac{(\sigma - \sigma_Y)}{E_T} \qquad \text{for } \sigma > \sigma_Y \;.$$

The tube is again subjected to a combined axial force F and torque T, but now $T = \alpha RF$ where α is a constant.

(i) Determine the yield torque T_Y under this combined loading. [25%]

(ii) Obtain the plastic strain components in the tube at $T = 2T_Y$. [40%]

END OF PAPER

ENGINEERING TRIPOS PART IIB

Module 4C9 Continuum Mechanics

Data sheet

Suffix notation

A repeated suffix implies summation

$\boldsymbol{a} = a_i \boldsymbol{e}_i \qquad \boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$

$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ can be written as $c_i = e_{ijk} a_j b_k$

$\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$ can be written as $A_{ij} = a_i b_j$

Kronecker delta: $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$

Note that $\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$

Permutation symbol: $e_{ijk} = 1$ when i, j, k are in cyclic order

$e_{ijk} = -1$ when i, j, k are in anti-cyclic order

$e_{ijk} = 0$ when any indices repeat

$e - \delta$ identity: $e_{ijk} e_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}$

$\text{grad}\,\phi = \nabla \phi = \phi_{,i} \boldsymbol{e}_i$

$\text{div}\,\boldsymbol{v} = \nabla \cdot \boldsymbol{v} = v_{i,i}$

$\text{curl}\,\boldsymbol{v} = \nabla \times \boldsymbol{v} = e_{ijk} v_{k,j} \boldsymbol{e}_i$

Gauss's theorem (the divergence theorem):

$$\int_V \frac{\partial A_{ij}}{\partial x_j} dV = \oint_S A_{ij} n_j \, dS$$

Stokes's theorem:

$$\int_S e_{ijk} \frac{\partial A_{pk}}{\partial x_j} n_i dS = \oint_C A_{pk} \; dx_k$$

Isotropic linear elasticity

Equilibrium: $\sigma_{ij,j} + b_i = 0 \quad , \quad \sigma_{ij} = \sigma_{ji}$

Compatibility: $\varepsilon_{ij,kp} + \varepsilon_{kp,ij} - \varepsilon_{pj,ki} - \varepsilon_{ki,pj} = 0$

Constitutive relationships: $\sigma_{ij} = \frac{E}{(1+\nu)}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$

Lame's constants: $\mu = G = \frac{E}{2(1+\nu)} \quad , \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$

At equilibrium the potential energy Π is minimised, i.e.:

$$\delta\Pi = \int_V \delta U dV - \int_S t_i^e \delta u_i dS - \int_V b_i \delta u_i dV = 0$$

The strain energy density U is given by: $\sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}$

Definitions: σ_{ij} is the stress tensor, ε_{ij} is the infinitesimal strain tensor, b_i is the body force vector, t_i^e is the external traction vector and u_i is the displacement vector.

Isotropic linear viscoelasticity

Relaxation modulus, $E_r(t)$:

if $\varepsilon(t) = \varepsilon_0 H(t)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$, then $\sigma(t) = \varepsilon_0 E_r(t)$

Creep compliance, $J_c(t)$:

if $\sigma(t) = \sigma_0 H(t)$, where $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$, then $\varepsilon(t) = \sigma_0 J_c(t)$

The Laplace transforms of $E_r(t)$ and $J_c(t)$ are related by: $\bar{E}_r(s)\,\bar{J}_c(s) = \frac{1}{s^2}$

Boltzmann superposition principle:

$$\sigma(t) = \int_0^t \frac{\partial \varepsilon(\tau)}{\partial \tau} E_r(t-\tau) d\tau + \varepsilon(0) E_r(t)$$

$$\varepsilon(t) = \int_0^t \frac{\partial \sigma(\tau)}{\partial \tau} J_c(t-\tau) d\tau + \sigma(0) J_c(t)$$

Correspondence principle: in the Laplace domain, the viscoelastic solution corresponds to the elastic solution, with the substitution $E \rightarrow s\bar{E}_r(s)$

J2 flow theory

The von Mises effective stress is $\sigma_e = \left(\frac{3}{2} s_{ij} s_{ij} \right)^{1/2}$

in terms of the deviatoric stress $s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}$

The von Mises effective strain rate is $\dot{\varepsilon}_e = \left(\frac{2}{3} \dot{\varepsilon}_{ij}^{\mathrm{PL}} \dot{\varepsilon}_{ij}^{\mathrm{PL}} \right)^{1/2}$ in terms plastic strain rate $\dot{\varepsilon}_{ij}^{\mathrm{PL}}$

Yield condition: $f = \sigma_e - Y \le 0$ where Y is the current yield strength, equal to the uniaxial yield strength at a plastic strain of ε_e.

The plastic strain rate satisfies normality: $\dot{\varepsilon}_{ij}^{\mathrm{PL}} = \frac{3}{2} \frac{s_{ij}}{\sigma_e} \dot{\varepsilon}_e$

where $\dot{\varepsilon}_e = \frac{\dot{\sigma}_e}{h}$ and h is the slope of the uniaxial curve of stress versus plastic strain.

Slip line field theory

Hencky equilibrium equations:

$p + 2k\phi = \text{constant}$ on an α -line

$p - 2k\phi = \text{constant}$ on a β -line

Geiringer kinematic equations:

$du - v d\phi = 0$ on an α -line

$dv + u d\phi = 0$ on a β -line

where u and v are the components of velocity along the α and β lines, respectively.

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