Version GNW/3

EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 30 April 2019 2 to 3.40

Module 4C9

CONTINUUM MECHANICS

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4C9 datasheet (3 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 A large viscoelastic block is loaded by a uniformly distributed pressure p acting over a small circular patch of radius a, as shown in Fig. 1(a). The viscoelastic properties of the material can be modelled using the time dependent relaxation modulus

$$E_r(t) = E e^{-\frac{Et}{\eta}},$$

where E and η are material constants. For 3D deformations, the Poisson's ratio ν can be assumed to be time-independent.

(a) If the block were *linear elastic*, then the non-zero stress components at the centre of the loading patch would be

$$\sigma_{11} = \sigma_{22} = -\frac{1+2\nu}{2}\frac{p}{\pi a^2}, \quad \sigma_{33} = -\frac{p}{\pi a^2} \tag{1}$$

(i) Briefly explain what is meant by the correspondence principle, and why it can be applied to this problem. [10%]

(ii) Hence explain why, if the viscoelastic block were subjected to a time-dependent pressure $p(t)$ acting over the circular patch, the expressions for the stress components given in eq. (1) would be applicable, but with the substitution $p = p(t)$. [5%]

(b) Starting with the corresponding *linear elastic* constitutive relationship, or otherwise, show that the time variation in strain component ε_{33} at the centre of the loading patch for the *viscoelastic material* is given by

$$\varepsilon_{33}(t) = \int_0^t J_c(t-\tau)\frac{\partial}{\partial\tau}\left[\sigma_{33}(\tau) - \nu\sigma_{11}(\tau) - \nu\sigma_{22}(\tau)\right] d\tau,$$

where $J_c(t)$ is the creep compliance and τ is a dummy time variable. [20%]

(c) For the pressure-time history $p(t)$ shown in Fig. 1(b):

(i) Derive expressions for $\varepsilon_{33}(t)$ at the centre of the loading patch for times $t \le t_0$ and $t > t_0$. [40%]

(ii) Derive an expression for the additional pressure that would need to be applied to the circular loading patch during the period $t > t_0$ in order to hold $\varepsilon_{33}(t)$ at a constant value, given by $\varepsilon_{33}(t_0)$. [25%]

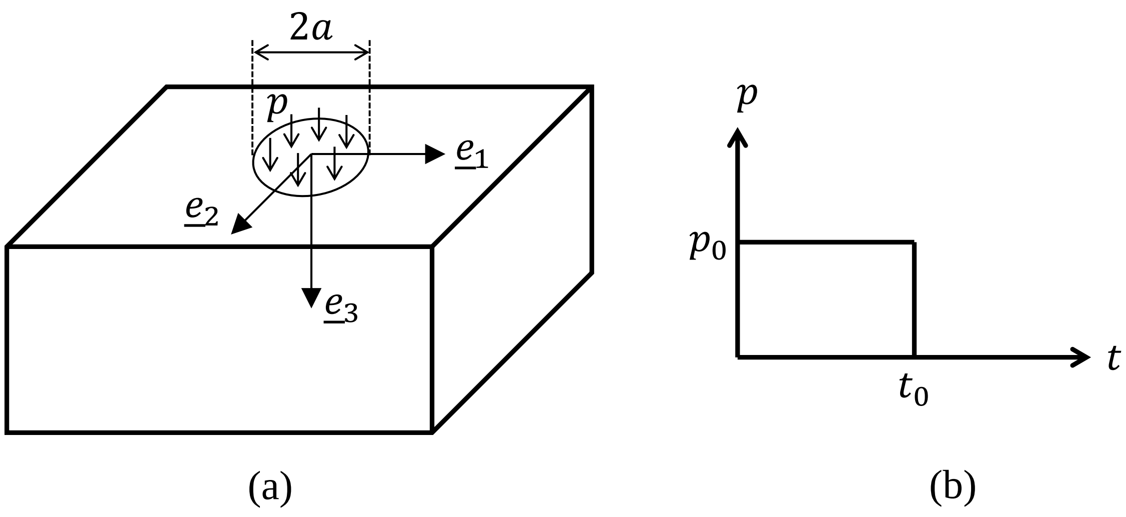

Fig. 1

(TURN OVER

2 (a) If $\boldsymbol{u}(\boldsymbol{x})$ is a vector field and $\phi(\boldsymbol{x})$ is a scalar field, both functions of position $\boldsymbol{x}$, use indicial notation to prove the following identities.

(i) $\nabla \cdot (\phi \boldsymbol{u}) = \phi \nabla \cdot \boldsymbol{u} + (\nabla \phi) \cdot \boldsymbol{u}$ [5%]

(ii) $\nabla \cdot \nabla \times \boldsymbol{u} = 0$ [10%]

(iii) $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla (\nabla \cdot \boldsymbol{u}) - \nabla \cdot \nabla \boldsymbol{u}$ [10%]

(b) An adhesive is modelled as a linear elastic layer of thickness H perfectly bonded to a pair of flat rigid plates, as shown in Fig. 2. The strain energy density in the elastic layer

$$U = \frac{E}{2(1+\nu)} \left[\varepsilon_{ij}\varepsilon_{ij} + \frac{\nu}{1-2\nu} \varepsilon_{ii}\varepsilon_{jj} \right],$$

where ε_{ij} are the components of the infinitesimal strain tensor, E is the Young's modulus and ν is the Poisson's ratio. Due to curing effects, the Young's modulus varies with distance x_1 in the $\boldsymbol{e}_1$ direction, i.e. $E = E(x_1)$. The Poisson's ratio is constant.

The bottom rigid plate is fixed (zero displacement or rotation). The top rigid plate is displaced $\boldsymbol{u} = w\boldsymbol{e}_1$ (other displacements and rotations zero). The elastic layer can be considered infinitely large in the $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ directions, and so $\varepsilon_{22} = \varepsilon_{33} = 0$.

(i) Using the method of minimum potential energy, derive the differential equation governing the displacement field $\boldsymbol{u}(\boldsymbol{x})$ within the elastic layer, stating any boundary conditions. Body forces can be neglected. [50%]

(ii) For the case $E = E_0 \exp(-x_1/H)$, where E_0 is a constant, derive expressions for the displacement field $\boldsymbol{u}(\boldsymbol{x})$ in the elastic layer and the force per unit area acting on the top plate. [25%]

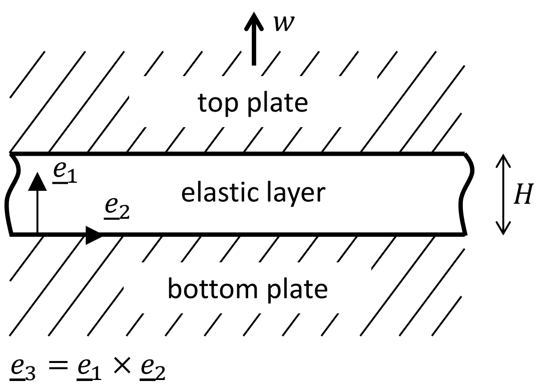

Fig. 2

3 (a) By expanding for three-dimensions, show that $e_{ijk}\sigma_{jk} = 0$ if $\sigma_{ij} = \sigma_{ji}$. [10%]

(b) Show that equilibrium of a body Ω in the absence of inertia effects requires that $-\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{f}$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, where $\boldsymbol{\sigma}$ is the Cauchy stress and $\boldsymbol{f}$ is a force per unit volume. [20%]

(c) The velocity gradient is given by $\boldsymbol{l} = \nabla \boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity field. Show that $\boldsymbol{l} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$, where $\boldsymbol{F}$ is the deformation gradient. [10%]

(d) The stress power P is given by:

$$P = \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{d} \, \mathrm{d}v = \int_{\Omega_0} \boldsymbol{\tau} : \boldsymbol{d} \, \mathrm{d}V = \int_{\Omega_0} \boldsymbol{S} : \dot{\boldsymbol{E}} \, \mathrm{d}V,$$

where $\boldsymbol{d} = (\boldsymbol{l} + \boldsymbol{l}^T)/2$ is the rate of deformation, Ω is the spatial domain, Ω_0 is the reference (material) domain and $\boldsymbol{E} = \left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right)/2$. In terms of the Cauchy stress and the deformation gradient, give expressions for:

(i) The Kirchhoff stress $\boldsymbol{\tau}$. [10%]

(ii) The second Piola–Kirchhoff stress $\boldsymbol{S}$. [20%]
Hint: $\boldsymbol{A} : (\boldsymbol{B}\boldsymbol{C}) = (\boldsymbol{B}^T\boldsymbol{A}) : \boldsymbol{C} = (\boldsymbol{A}\boldsymbol{C}^T) : \boldsymbol{B}$.

(e) Objectivity of a second-order tensor $\boldsymbol{A}$ requires that it transforms according to $\tilde{\boldsymbol{A}} = \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T$, where $\boldsymbol{Q}$ represents a rigid body rotation.

(i) Show that the simple time derivative of the Cauchy stress, $\dot{\boldsymbol{\sigma}}$, is not objective. [10%]

(ii) Noting that the deformation gradient transforms according to $\tilde{\boldsymbol{F}} = \boldsymbol{Q}\boldsymbol{F}$, show that the rate of deformation tensor $\boldsymbol{d} = (\boldsymbol{l} + \boldsymbol{l}^T)/2$, where $\boldsymbol{l} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$, is objective. [20%]

END OF PAPER

THIS PAGE IS BLANK