EGT3
ENGINEERING TRIPOS PART IIB

Friday 29 April 2016   2 to 3.30

Module 4F2

ROBUST AND NONLINEAR SYSTEMS AND CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 An industrial process is modelled by the transfer function

$$G(s) = \alpha \frac{e^{-sD}}{s+1}$$

where $D \geq 0$ is a given transport delay and $1 \leq \alpha \leq 2$. In what follows you will design proportional controllers $K(s) = k$ that guarantee closed loop stability.

(a) Consider the closed loop in Figure 1.

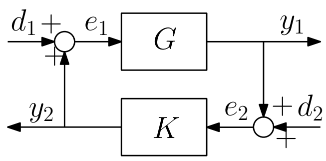

Fig. 1

(i) Compute the smallest $\gamma_1 > 0$ such that $\sup_{1\leq\alpha\leq 2} \|G\|_\infty \leq \gamma_1$. [20%]

(ii) Using the small gain theorem and the bound γ_1, find the range of gains k that guarantee asymptotic stability of the closed loop. [20%]

(b) Consider $D = 0$. For a more advanced design the uncertainty on the system is modelled as a multiplicative uncertainty $G(s) = G_0(s)(1+\Delta)$ where $G_0(s) = \frac{1}{s+1}$ and $\|\Delta\|_\infty \leq 1$.

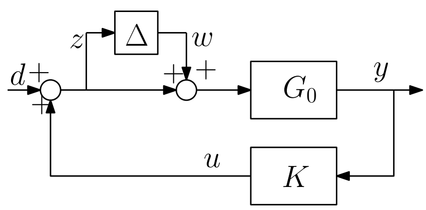

Fig. 2

(i) Determine $\|T_{w\to z}\|_\infty$ as a function of k, where $T_{w\to z}(s)$ is the nominal (i.e. $\Delta(s) = 0$) transfer function relating $\overline{w}(s)$ to $\overline{z}(s)$. [25%]

(ii) Using the small gain theorem, find the range of gains k that guarantee robust stability. [20%]

(iii) Consider an extended set of perturbations $\|\Delta\|_\infty \leq \rho$, where $\rho > 1$. Show that for any $\rho > 1$ there exists a proportional gain k that guarantees robust stability. [15%]

2 Consider the closed loop system in Figure 3. Let $W_1(s)$ and $W_2(s)$ be stable scalar transfer functions. In what follows $T_{a\to b}(s)$ denotes the transfer function from a given input $\bar{a}(s)$ to a given output $\bar{b}(s)$.

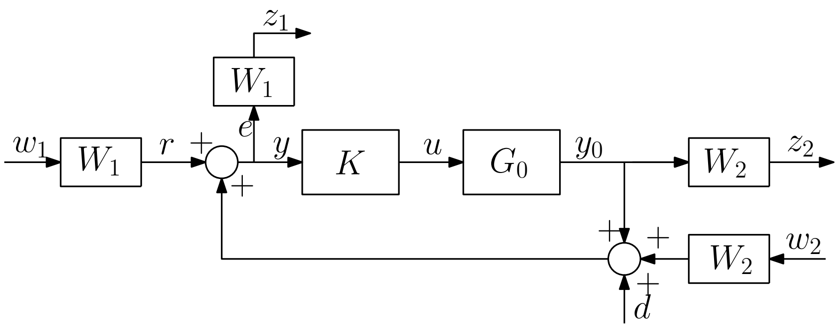

Fig. 3

(a) Take $\|\Delta\|_\infty \leq 1$ and $W_2(s) = 10$. Suppose that $\|W_2 T_{d\to y_0} W_2\|_\infty < 1$.

(i) Show that $K(s)$ guarantees robust stability for $G(s) = (I + 100\Delta(s))G_0(s)$. [20%]

(ii) Show that $K(s)$ guarantees nominal performance $\|T_{d\to y_0}\|_\infty < \frac{1}{100}$ and $\|T_{r\to e}\|_\infty > \frac{99}{100}$. [20%]

(b) (i) Draw an equivalent block diagram to Figure 3 in which there is a "generalised plant" P with inputs and outputs

$$\begin{bmatrix} w_1 \\ w_2 \\ d \\ u \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_1 \\ z_2 \\ e \\ y \end{bmatrix}$$

[20%]

(ii) Define the robust performance problem and briefly discuss a technique to address it. [20%]

(iii) Take $W_1(s) = \alpha > 0$ and $W_2(s) = \beta > 0$. Show that there is no controller K that can achieve $\|T_{w_1\to z_1}\|_\infty < 1$ and $\|T_{w_2\to z_2}\|_\infty < 1$ if $\frac{1}{\alpha^2} + \frac{1}{\beta^2} \leq 1$. [20%]

3 Consider the (positive or negative) feedback interconnection of a *stable* linear system, with transfer function $H(s)$, and a (saturation) static nonlinearity $y = \varphi(u)$ described by

$$\varphi(u) = \begin{cases} -1, & u \leq -1 \\ u, & -1 \leq u \leq 1 \\ +1, & u \geq 1 \end{cases}$$

(a) Suppose that $H(s) = \frac{2}{s+1}$. Determine the equilibria of the feedback system and their stability, both in the positive and in the negative feedback configuration. Briefly discuss the qualitative behaviour of both systems. [30%]

(b) Draw a phase-portrait of the *positive* feedback system for the transfer function

$$H(s) = \frac{2}{\varepsilon s^2 + s + 1}$$

where $\varepsilon > 0$ is a small parameter. Sketch on the phase portrait the stable manifold of the unstable equilibrium and explain why it plays an important role in organizing the global behaviour. [30%]

(c) Propose a candidate stable transfer function $H(s)$ for the *negative* feedback system to possess a limit cycle oscillation. Briefly explain how one could predict the period and amplitude of this oscillation. [20%]

(d) Generalize your analysis in (a) to an *arbitrary* stable transfer function $H(s)$ that satisfies $H(0) > 1$. [20%]

4 Consider the (Duffing) nonlinear second-order mechanical model

$$m\ddot{x} + \frac{dE}{dx} + k\dot{x} = u \tag{1}$$

for the potential energy $E(x) = \frac{x^4}{4} + \alpha\frac{x^2}{2}$, where u is an external force. The total energy of the free (i.e. $u = 0$) system is

$$V(x, \dot{x}) = E(x) + \frac{m\dot{x}^2}{2} \tag{2}$$

(a) Evaluate the time-derivative of V along the trajectories. [10%]

(b) Sketch plots of the potential energy for $\alpha \geq 0$ and for $\alpha < 0$ and use these plots to describe the behaviour qualitatively. [20%]

(c) Assume $u = 0$ and $\alpha \geq 0$. Prove that the equilibrium $(x, \dot{x}) = (0, 0)$ is Lyapunov stable when $k = 0$ and asymptotically stable when $k > 0$. For $k > 0$, what is the basin of attraction of the equilibrium? [20%]

(d) (i) Show that an output can be selected so that the mechanical system (1) is passive when $\alpha \geq 0$. What is the physical interpretation of the storage function and of the supply rate? [10%]

(ii) Show that the transfer function $H(s) = k_p + \frac{k_i}{s}$ defines a strictly passive system. Discuss the implication of that property for feedback control of the Duffing system. [10%]

(e) (i) Prove that the equilibrium $(x, \dot{x}) = (0, 0)$ is unstable when $u = 0$ and $\alpha < 0$. [10%]

(ii) Sketch the phase portrait of the system near this unstable equilibrium. [10%]

(iii) Explain why this equilibrium is called a saddle point and why it plays an important role for the global behaviour of the system when the damping coefficient $k > 0$ is small. [10%]

END OF PAPER

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