EGT3
ENGINEERING TRIPOS PART IIB

Monday 4 May 2015   2 to 3.30

Module 4F3

OPTIMAL AND PREDICTIVE CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4F3 Optimal and Predictive Control data sheet (2 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Consider a linear discrete-time plant with state $x(k)$ and input $u(k)$, described as

$$x(k+1) = Ax(k) + Bu(k).$$

Given matrices $Q \geq 0$, $P \geq 0$ and $R > 0$, a particular unconstrained receding horizon controller minimises the following *cost function*

$$J(x(k), \mathbf{u}) = x_N^T P x_N + \sum_{i=0}^{N-1} \left(x_i^T Q x_i + u_i^T R u_i \right) \qquad \textbf{(C1)}$$

where x_i and u_i are, respectively, the predicted states and inputs at time $k+i$, computed from measurement $x(k)$ and $\mathbf{u} \triangleq [u_0^T, u_1^T, \ldots, u_{N-1}^T]^T$.

(a) Explain the term *Control Lyapunov Function* for this class of systems. [20%]

(b) (i) Assume that A has all eigenvalues inside the unit disc. Show that the *Value Function* can be used to prove stability of the closed loop system, if P is chosen to satisfy the discrete-time Lyapunov Equation [40%]

$$A^T P A - P = -Q.$$

(ii) How should you choose P to guarantee closed-loop stability if A has at least one eigenvalue outside the unit disc? [15%]

(c) An alternative predictive controller solves the following optimisation problem

$$\min_{u_0, \ldots, u_{N_1-1}} J(x(k), \mathbf{u}) = \sum_{i=0}^{N_1-1} \left(x_i^T Q x_i + u_i^T R u_i \right) + \sum_{i=N_1}^{N_2} x_i^T Q x_i \qquad \textbf{(C2)}$$

where N_1 is termed the *control horizon* and N_2 is termed the *prediction horizon*, and $u_i = 0$ for all $i \geq N_1$. Verify that **C1**, with A, Q and P related in the same way as in part (b)(i), is equivalent to **C2** with $N_2 = \infty$. [25%]

2 (a) Predictive control allows constraints to be included in the controller design.

(i) With reference to two practical examples, explain the relevance of state constraints in control. [15%]

(ii) Comment on the *computational* advantages and disadvantages of predictive control for systems with constraints. Consider what must be computed both during the initial design, and when the controller is in operation on the plant. [15%]

(b) A constrained predictive tracking controller solves the following optimisation problem at each time step

$$\min_{\substack{u_0,\ldots,u_{N-1}\\ x_0,\ldots,x_N}} \quad \frac{1}{2}(x_N - x_s)^T P(x_N - x_s) + \frac{1}{2}\sum_{i=0}^{N-1}\Big((x_i - x_s)^T Q(x_i - x_s) + (u_i - u_s)^T R(u_i - u_s)\Big)$$

$$\begin{aligned} \text{subject to:} \quad x_0 &= x(k) & \\ x_{i+1} &= Ax_i + Bu_i + d & i = 0,\ldots,N-1 \\ Mx_i + Eu_i &\le b & i = 0,\ldots,N-1 \\ M_N x_N &\le b_N & \end{aligned}$$

where $x(k)$ is the state measured at time k, and x_i and u_i are the predicted state and input, respectively, at time $k+i$. The target state and input are x_s and u_s respectively. A and B define the prediction model, d is a constant and known disturbance, and M, E, M_N, b, b_N define a non-empty constraint set. Let $N = 2$ and consider a decision variable $\underline{\theta}$ of interleaved predicted inputs and states, defined as $\underline{\theta} = [u_0^T, x_1^T, u_1^T, x_2^T]^T$. Find matrices H, h, G, g, F and f for the equivalent constrained quadratic program in the standard form [35%]

$$\begin{aligned} \min_{\underline{\theta}} \quad & \frac{1}{2}\underline{\theta}^T H \underline{\theta} + h^T \underline{\theta} \\ \text{subject to:} \quad & G\underline{\theta} \le g \quad \text{and} \quad F\underline{\theta} = f. \end{aligned} \qquad \textbf{(QP)}$$

(c) For the output $y = Cx + Du$ to track a constant reference r, the target equilibrium state x_s and input u_s should satisfy the equalities

$$\begin{aligned} Ax_s + Bu_s + d &= x_s \\ Cx_s + Du_s &= r. \end{aligned}$$

Even if the appropriate rank conditions hold to guarantee a solution in the absence of inequality constraints, it still cannot be guaranteed that there exists a pair (x_s, u_s) that satisfies the state and input constraints from part (b) for abitrary r.

Construct a constrained quadratic program of the form (**QP**) that finds a feasible equilibrium pair and reference $\hat{r}$ that minimises $\|\hat{r} - r\|_2^2$. [35%]

3 (a) Consider a continuous-time linear system $G(s)$, with input w and output y.

(i) How does the $\mathcal{H}_\infty$ norm of $G(s)$, $\|G(s)\|_\infty$ relate w and y? Define any additional terms used and discuss their interpretation. [20%]

(ii) Give an expression for the computation of $\|G(s)\|_\infty$ that involves only $G(s)$, and discuss its interpretation. [10%]

(b) Consider the generalised plant $P(s)$ described in state-space form as

$$\dot{x} = Ax + \begin{bmatrix} B_1 & 0 \end{bmatrix} w + B_2 u$$
$$z = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = C_2 x$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Design a stabilising static control law of the form $u = Ky$ such that $\|\mathcal{F}_l(P(s), K)\|_\infty \leq \gamma$.
Hint: Let $\alpha = (1 - \gamma^{-2})$ to simplify algebraic manipulations. [40%]

(c) (i) Assume you are provided with a numerical algorithm that can determine if a stabilising K exists for a given value of γ, and if so, compute its numerical value. Using this algorithm, describe how you would find the minimum achievable value of $\|\mathcal{F}_l(P(s), K)\|_\infty$. [15%]

(ii) Justify (in control theoretical terms) why you should not attempt to adapt your design of part (b) to the case where $C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and all other matrices are unchanged. Demonstrate any assertions made. [15%]

4 (a) (i) What is meant by a convex function? [10%]

(ii) Sketch an example of a non-convex set in two dimensions, graphically emphasising the characteristic that makes it non-convex. [10%]

(iii) Why is convexity a useful property for an optimisation problem? [10%]

(b) Consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k$$

and the discrete-time finite-horizon linear quadratic regulator that minimises

$$J(x_0, u_0, u_1) = x_2^T X_2 x_2 + \sum_{k=0}^{1} \left(x_k^T Q x_k + u_k^T R u_k + x_k^T S u_k + u_k^T S^T x_k \right).$$

(i) State conditions sufficient for this cost function to be convex (assume $S \neq 0$). [10%]

Consider the system matrices, and cost weighting matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 5, \quad S = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 100 & 0 \\ 0 & 20 \end{bmatrix}.$$

(ii) Compute the numerical value of the control gain at time $k = 1$. [20%]

(iii) State an expression for the value function $V(x,k)$ at time $k = 2$ and $k = 1$ as a function of x and compute $V(x,1)$ if $x = [1,1]^T$. [20%]

(c) Consider an ideal situation, with neither model uncertainty nor any other disturbances, and where computation is instantaneous. For an arbitrary initial state, explain why the trajectory of the closed-loop system over two time steps using the controller from part (b) might be different from that obtained using an unconstrained predictive controller with the same cost function. [20%]

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4F3: Optimal and Predictive Control Numerical Answers (2015)

E. N. Hartley

1. (a) $V(0) = 0$, $V(x) > 0$ for $x \neq 0$, $\exists u = \kappa(x)$ such that $V(Ax + Bu) \leq V(x)$.

   (b)(i)

$$\begin{aligned} V(x_1^*) &\leq J(x_1^*, \tilde{\mathbf{u}}) \\ J(x_1^*, \tilde{\mathbf{u}}) &= V(x_0) - x_0^{*T} Q x_0^* - u_0^{*T} R u_0 - x_N^{*T} P x_N^* + x_N^{*T} Q x_N^* + x_N^{*T} A^T P A x_N^* \\ &= V(x_0) - x_0^{*T} Q x_0^* - u_0^{*T} R u_0 + \underbrace{x_N^{*T}(A^T P A - P + Q) x_N^*}_{0} \\ &= V(x_0) - x_0^{*T} Q x_0^* - u_0^{*T} R u_0 \leq V(x_0) \end{aligned}$$

   (ii) $(A + BK)^T P (A + BK) - P = -Q - K^T R K$

   (c) $\sum_{i=N_1}^{\infty} x_i^T Q x_i = \sum_{i=N_1}^{\infty} \left(x_i^T P x_i - x_i A^T P A x_i \right) = \lim_{i \to \infty} x_{N_1}^T P x_{N_1} + x_i^T P x_i = x_{N_1}^T P x_{N_1}$ since $\rho(A) < 1$

2. (a) (i) – (ii) –

   (b)

$$H = \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & P \end{bmatrix}, \quad h = \begin{bmatrix} -Ru_s \\ -Qx_s \\ -Ru_s \\ -Px_s \end{bmatrix}, \quad F = \begin{bmatrix} B & -I & 0 & 0 \\ 0 & A & B & -I \end{bmatrix}, \quad f = \begin{bmatrix} -Ax(k) - d \\ -d \end{bmatrix}$$

$$G = \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & M & E & 0 \\ 0 & 0 & 0 & M_N \end{bmatrix}, \quad g = \begin{bmatrix} -Mx(k) + b \\ b \\ b_N \end{bmatrix}$$

   (c)

$$\underline{\theta} = \begin{bmatrix} x_s \\ u_s \\ \hat{r} \end{bmatrix}, \quad F = \begin{bmatrix} (A - I) & B & 0 \\ C & D & -I \end{bmatrix}, \quad f = \begin{bmatrix} -d \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} M & E & 0 \\ M_N & 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} b \\ b_N \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & & \\ & 0 & \\ & & I \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix}.$$

3. (a)(i) $\|y\|_2 \leq \|G\|_\infty \|w\|_2$, $\|y\|_2 = \sqrt{\int_{-\infty}^{\infty} y(t)^T y(t)\, \mathrm{d}t}$ (energy in signal)

   (ii) $\|G\|_\infty = \sup_\omega \overline{\sigma}(G(j\omega))$ (largest magnitude of the gain of the system over all frequencies)

   (b) $u = -B_2^T X x = -\begin{bmatrix} \alpha^{-1/2} & \sqrt{2}\alpha^{-3/4} \end{bmatrix} x = -\begin{bmatrix} (1-\gamma^{-2})^{-1/2} & \sqrt{2}(1-\gamma^{-2})^{-3/4} \end{bmatrix} x.$

   (c) Use interval bisection.

   (d) $\begin{bmatrix} C_2 \\ C_2 A \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \implies$ not full rank so not observable.

4. (a)(i) $f((1-\alpha)x + \alpha y) \leq (1-\alpha) f(x) + \alpha f(y)$, for $\alpha \in [0, 1]$

   [or equivalently, $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$],

   (ii) Two points in a shape connected by straight line exiting shape.

   (iii) Local = global minimum. Efficient solution.

   (b) (i) $X_2 \geq 0$, $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0$.

   (ii) $u_1 = -\begin{bmatrix} 1 & 7/5 \end{bmatrix} x_1$.

   (iii) $x^T X_2 x$, 134.

   (c) DLQR $\implies$ apply whole input sequence, MPC $\implies$ receding horizon: recompute sequence at each time step.