EGT4
ENGINEERING TRIPOS PART IIB

Monday 18 April 2016 2 to 3:30

Module 4F3

OPTIMAL AND PREDICTIVE CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4F3 data sheet.
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Consider the active shock absorber in Figure 1. The dynamics of the system satisfies the balance of forces $ma + kp + cv = f_h$, where a is the acceleration of the mass m, p is the displacement of the mass, v is the velocity of the mass, f_h is the force exerted by a hydraulic actuator, k is the spring constant, and c is the damping coefficient. The actuator dynamics are approximated by a first order lag $\tau\left(\frac{d}{dt}f_h\right) = -f_h + u$ with time constant $\tau > 0$ and input u.

(a) (i) Given the state vector $x = \begin{bmatrix} p & v & f_h \end{bmatrix}^T$ and measured output $y = p$, show that the overall system behaviour is characterised by the linear equations $\dot{x} = Ax + Bu$, $y = Cx + Du$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k}{m} & -\frac{c}{m} & \frac{1}{m} \\ 0 & 0 & -\frac{1}{\tau} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = 0\,. \qquad (1)$$

[5%]

(ii) Show that this system is stable. [10%]

(b) Let Q be the infinite-time observability gramian: $Q = \int_0^\infty e^{A^T t} C^T C e^{At} dt$.

(i) Show that Q is a solution to the Lyapunov equation $A^T Q + QA + C^T C = 0$. [20%]

(ii) If $Q > 0$, show that (A, C) is observable (without computing the observability matrix). [20%]

(c) (i) What is meant by a *balanced realisation* of a transfer function, and by its *Hankel singular values*? [15%]

(ii) Explain how a balanced realisation of a stable system can be used to obtain lower-order approximations of the system. [15%]

(iii) Let $G(s) = C(sI - A)^{-1}B$, where A, B, C are defined in (1) for some particular parameter values. $\hat{G}(s)$ is a second-order approximate model of $G(s)$, obtained from a balanced realisation. Figure 2 shows the Bode magnitude plot of $E(s) = G(s) - \hat{G}(s)$. Estimate $\|E\|_\infty$, and hence obtain lower and upper bounds for the third Hankel singular value of $G(s)$. [15%]

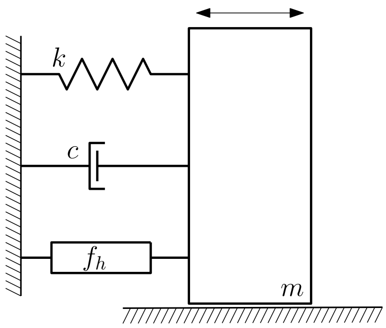
k
c
f_h
m

Fig. 1

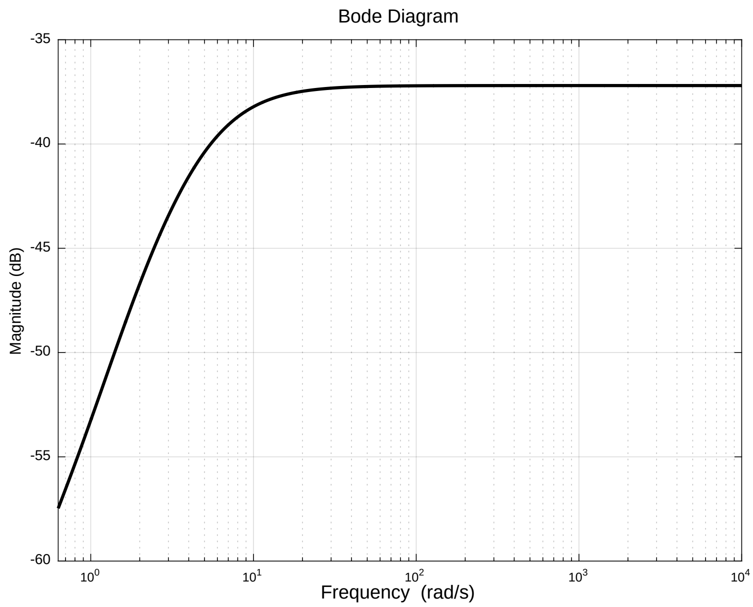
Bode Diagram
Magnitude (dB)
-35
-40
-45
-50
-55
-60
10^0
10^1
10^2
10^3
10^4
Frequency (rad/s)

Fig. 2

2 A mass-spring-damper system with a unit mass is described by the state-space equations

$$\dot{x} = Ax + Bu + Bw = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \quad y = Cx = \begin{bmatrix} 1 & 0 \end{bmatrix} x \qquad (2)$$

where $u = Kx = -[k, c]x$ is the force acting on the mass, y is the position of the mass, k is the spring stiffness and c is the damper coefficient.

(a) Take $w = 0$ and consider the cost

$$\int_0^\infty [x(t)^T Qx(t) + u(t)^T Ru(t)]dt, \quad \text{where} \quad Q = C^T C \text{ and } R = 1\ . \qquad (3)$$

(i) Show that the values of k and c which minimise the cost (3) are given by $\begin{bmatrix} k & c \end{bmatrix} = B^T X$ for $X = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}$. [25%]

(ii) Noting that (A, B) is controllable and (A, C) is observable, what conclusion can be drawn about the stability of $(A + BK)$, if $K = -[k, c]$ is chosen as in part (i)? [10%]

(b) Let u and y be the input and output signal vectors of a system with transfer function $G(s)$.

(i) The infinity norm $\|G\|_\infty$ satisfies the relationship $\|G\|_\infty = \sup \|y\|_2 / \|u\|_2$ where the supremum is taken over all non-zero inputs with $\|u\|_2 < \infty$. How are the signal norms $\|u\|_2$ and $\|y\|_2$ defined? [10%]

(ii) If $V(x) = x^T X x$ for some $X = X^T > 0$, $G(s) = C(sI - A)^{-1}B$, and X satisfies the Riccati equation

$$A^T X + XA + C^T C + \frac{1}{\gamma^2} XBB^T X = 0$$

then it can be shown that

$$\frac{dV}{dt} + y^T y - \gamma^2 u^T u \leq 0. \qquad (4)$$

Show that, if (4) holds, then $\|G\|_\infty \leq \gamma$. [20%]

(iii) For the system defined in (2), let $H(s)$ be the transfer function from w to y. If the coefficients k and c are chosen as in part (a)(i), show that $\|H\|_\infty \leq 1$. [35%]

3 (a) (i) How is a *convex set* defined? [15%]

(ii) When is an optimisation problem convex? [15%]

(iii) Give two examples of convex optimisation problems. [15%]

(b) A plant with state x_k and input u_k at time k is described by the discrete-time state-space model $x_{k+1} = Ax_k + Bu_k$. A predictive controller minimises the cost function

$$V(x_0, u_0, u_1, \ldots, u_{N-1}) = x_N^T P x_N + \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k)$$

subject to the constraints $Mx_k + Eu_k \leq b$ for $k = 0, 1, \ldots, N-1$. Show that this problem can be written as a standard quadratic programming problem of the form

$$\text{minimise } \theta^T H \theta \quad \text{subject to} \quad F\theta - f = 0 \quad \text{and} \quad G\theta - g \leq 0$$

for suitable matrices F, G, H and suitable vectors f, g, with the vector θ containing the decision variables $u_0, u_1, \ldots, u_{N-1}$ and $x_1, x_2, \ldots, x_N$. [40%]

(c) A 'terminal constraint' of the form $M_N x_N \leq b_N$ is sometimes added to predictive control problems. Comment briefly (without technical details) on the reason for adding such a constraint, and on the properties that it should satisfy. [15%]

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4 An unconstrained predictive controller determines the input signal $u(k)$ at time k by minimising the cost function

$$V(x(k),\mathbf{u}) = x_N^T P x_N + \sum_{i=0}^{N-1} (x_i^T Q x_i + u_i^T R u_i)$$

where $x(k)$ is the measured current state, $x_0 = x(k)$,

$$x_{i+1} = Ax_i + Bu_i \quad \text{for} \quad i = 0, 1, \ldots, N-1$$

and setting $u(k) = u_0^*$, where the minimising input sequence is

$$\mathbf{u}^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*).$$

(a) Explain why repeated determination of the input signal in this manner results in a feedback system. [10%]

(b) What is meant by the phrase *terminal cost* in this context? [10%]

(c) Assume that $P > 0$, $Q > 0$ and $R > 0$, and that K is a matrix such that all the eigenvalues of $A + BK$ lie within the unit circle. Show that, if

$$(A+BK)^T P(A+BK) - P \leq -Q - K^T RK$$

then the origin ($x = 0$) of the closed-loop system with the predictive controller is asymptotically stable. [50%]

(d) A particular 1-input, 1-state system has $A = 1.2$, $B = 1$, and the control system designer chooses $Q = 5$ and $R = 2$. Find a P which results in an asymptotically stable closed-loop system. [30%]

END OF PAPER

ANSWERS

Q.1(C)(iii): $0.0067 \leq \sigma_3 \leq 0.0133$ (but $0.007 \leq \sigma_3 \leq 0.014$ is acceptable).

Q.4(d): There is no unique answer to this question. K must be chosen in the range $-2.2 < K < -0.2$. Once a K is chosen in this range, P must be chosen to be larger than the value given in the following table:

K	-2.1	-2.0	-1.9	-1.8	-1.7	-1.6	-1.5	-1.4	-1.3	-1.2
P	72.74	36.11	23.96	17.94	14.37	12.05	10.44	9.29	8.46	7.88
K	-1.1	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	
P	7.49	7.29	7.27	7.48	7.97	8.94	10.78	14.78	27.26	