Version GV/4

EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 1 May 2019 9.30 to 11.10

Module 4F3

AN OPTIMISATION BASED APPROACH TO CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4F3 data sheet (two pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 The motion of an inkjet printer head is to be optimised. The printer head moves accordingly to the law

$$x_{k+1} = x_k + u_k$$

where x_k is the current position, x_{k+1} is the next position, and u_k is the offset.

(a) Let r be the desired target position and consider the finite horizon cost

$$J_1 = (x_T - r)^2 + \sum_{k=0}^{T-1} (x_k - r)^2 \qquad \text{for } T = 2\,.$$

(i) Using dynamic programming, find the optimal cost J_1^*, optimal trajectory x_k^*, and optimal (unconstrained) input sequence u_k, from the generic initial state $x_0^* = x_0$. [30%]

(ii) Explain why the modified cost

$$J_2 = (x_T - r)^2 + \sum_{k=0}^{T-1} \left((x_k - r)^2 + u_k^2\right) \qquad \text{for } T = 2$$

gives a smoother optimal motion x_k^* than J_1. Does J_2 guarantee that $x_T^* = r$? [10%]

(b) Consider the cost

$$J_3 = (x_T - r)^2 \qquad \text{for } T = 2\,.$$

Find the optimal input sequence constrained to $-1 \leq u_k^* \leq 1$, optimal trajectory x_k^*, and optimal cost J_3^*, for the initial condition $x_0 = 0$ and target $r = 2$. [30%]

(c) A simple physical model of the printer head is given by the dynamics of a frictionless mass of unit weight, with position p, and velocity v

$$\dot{p}(t) = v(t) \qquad \dot{v}(t) = u(t)$$

where u is the external driving force. An appropriate cost is

$$J_4 = \int_0^\infty (p(t) - r)^2 + u(t)^2 \, dt\,.$$

By formulating the problem of minimising this cost as a *linear quadratic regulator* (LQR) problem, find the optimal (unconstrained) control $u(t)$. [30%]

2 (a) Consider the first order system with transfer function

$$G(s) = \frac{1}{s+1}\ .$$

(i) Describe the two methods of computing the $\mathcal{H}_\infty$ norm of a stable system, one in the frequency domain and the other in the time domain / state space. [15%]

(ii) Compute the $\mathcal{H}_\infty$ norm of $G(s)$. [15%]

(b) Consider the associated generalised plant P_1 with realisation

$$\dot{x} = -x + u + w_1, \quad y = x + w_2, \quad z = \begin{bmatrix} x \\ u \end{bmatrix}$$

where w_1 and w_2 represent additive noise on state x and measured output y, respectively. z is the performance output and u is the control input.

(i) Define the $\mathcal{H}_\infty$ optimal control problem for the generalised plant P_1. [10%]

(ii) Derive state-feedback and output-feedback $\mathcal{H}_\infty$ controllers which guarantee that the closed loop transfer function from w to z has $\mathcal{H}_\infty$ norm smaller than or equal to 1. [40%]

(c) Consider the simplified generalised plant P_2 with realisation

$$\dot{x} = -x + u + w, \quad z = y = x$$

where w represents additive noise. Using linear matrix inequalities, describe how to find the state-feedback $\mathcal{H}_\infty$ controller $u = Kx$ which minimises the $\mathcal{H}_\infty$ norm of the closed loop transfer function from w to z. [20%]

(TURN OVER

3 (a) List some advantages and disadvantages of predictive control, referring to one or more different application areas in your answer. [25%]

(b) Write down the standard form of a *quadratic programming* (QP) optimisation problem. [10%]

(c) Consider a linear discrete time dynamical system

$$x_{k+1} = Ax_k + Bu_k$$

Predictive control is to be applied to this system with a receding horizon cost function

$$\sum_{i=k}^{k+N-1} \left(x_{i+1}^T Q x_{i+1} + u_i^T R u_i \right)$$

and constraints

$$|u_k| \leq U$$

for some U and all k. Assuming that the full state vector x_k can be measured at each step, show how the problem of choosing the control input at each step can be written as a QP problem. [40%]

(d) Explain how, and why, the cost function and constraints should be modified to ensure stability and feasibility. Equations are not required. [25%]

4 Consider the following three control problems. For each problem choose a suitable algorithm from the areas of Predictive Control, Reinforcement Learning or Optimal Control (a different algorithm for each problem). In each case describe the algorithm and its application to the problem in detail, explaining why it is appropriate to the problem. Included in your answers should be definitions of the terms *value function*, *policy iteration*, *value iteration* and the action-value function, Q. You may make reasonable assumptions.

(a) Control Problem 1: Determining the minimum fuel required, and the corresponding throttle settings to land a spacecraft on the surface of the moon from orbit. An accurate model of the spacecraft is available. You may assume that the spacecraft moves in one orbital plane (i.e. that the desired landing point is directly below the spacecraft at one point in its orbit) and that the orientation of the spacecraft can be controlled instantaneously. [33%]

(b) Control Problem 2: Maintaining straight and level flight for a damaged aircraft, where several control surfaces are damaged or unavailable but an accurate linearised mathematical model is available for both the aircraft and the available controls and it is known which controls are not available. [33%]

(c) Control Problem 3: An offline study into the possibility of controlling a heavily damaged aircraft, including being able to execute turns, where several control surfaces and one or more engines are unavailable, and parts of the wing are missing. An accurate simulation environment is available, incorporating important nonlinear characteristics from a detailed fluid dynamics model, and it is expected that an unconventional nonlinear control strategy would be required. [34%]

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Module 4F3: Optimal and Predictive Control

Data Sheet (available in the exam)

1. (a) For the dynamical system satisfying, $\dot{x} = f(x,u),\ x(0) = x_0$, and the cost function

$$J(x_0, u(\cdot)) = \int_0^T c(x(t), u(t))\, dt\ + J_T(x(T))$$

then under suitable assumptions the value function, $V(x,t)$, satisfies the Hamilton-Jacobi-Bellman PDE,

$$-\frac{\partial V(x,t)}{\partial t} = \min_{u \in U} \left(c(x,u) + \frac{\partial V(x,t)}{\partial x} f(x,u) \right), \quad V(x,T) = J_T(x).$$

(b) For $f(x,u) = Ax + Bu$, $c(x,u) = x^T Q x + u^T R u$, and $J_T(x) = x^T X_T x$, if $X(t)$ satisfies the Riccati ODE,

$$-\dot{X} = Q + XA + A^T X - XBR^{-1}B^T X, \quad X(T) = X_T,$$

then $J_{opt} = x_0^T X(0) x_0$ and $u_{opt}(t) = -R^{-1} B^T X(t) x(t)$.

2. For the discrete-time system satisfying $x_{k+1} = Ax_k + Bu_k$ with x_0 given and cost function,

$$J(x_0, u_0, u_1, \ldots, u_{h-1}) = \sum_{k=0}^{h-1} \left(x_k^T Q x_k + u_k^T R u_k \right) + x_h^T X_h x_h,$$

if X_k satisfies the backward difference equation,

$$X_{k-1} = Q + A^T X_k A - A^T X_k B (R + B^T X_k B)^{-1} B^T X_k A,$$

then $J_{opt} = x_0^T X_0 x_0$ and optimal control signal, $u_k = -(R + B^T X_{k+1} B)^{-1} B^T X_{k+1} A x_k$.

3. For the system satisfying,

$$\begin{bmatrix} \dot{x} \\ \hline z \\ y \end{bmatrix} = \left[\begin{array}{c|cc} A & \begin{bmatrix} B_1 & 0 \end{bmatrix} & B_2 \\ \hline \begin{bmatrix} C_1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ C_2 & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right] \begin{bmatrix} x \\ \hline w \\ u \end{bmatrix} \quad \text{where} \quad \begin{cases} (A, B_2) & \text{controllable} \\ (A, C_1) & \text{observable} \\ (A, B_1) & \text{controllable} \\ (A, C_2) & \text{observable} \end{cases}$$

(a) The optimal $\mathcal{H}_2$ controller is given by,

$$\begin{bmatrix} \dot{x}_k \\ \hline u \end{bmatrix} = \left[\begin{array}{c|c} A - B_2F - HC_2 & -H \\ \hline F & 0 \end{array}\right] \begin{bmatrix} x_k \\ \hline y \end{bmatrix}$$

where $F = B_2^T X$, $H = YC_2^T$, and X and Y are stabilising solutions to

$$0 = XA + A^T X + C_1^T C_1 - XB_2 B_2^T X \quad \text{(CARE)}$$

and

$$0 = YA^T + AY + B_1 B_1^T - YC_2^T C_2 Y \quad \text{(FARE)}$$

(b) The controller given by,

$$\begin{bmatrix} \dot{x}_k \\ \hline u \end{bmatrix} = \left[\begin{array}{c|c} \hat{A} - B_2F - HC_2 & -H \\ \hline F & 0 \end{array}\right] \begin{bmatrix} x_k \\ \hline y \end{bmatrix}$$

where $F = B_2^T X$, $H = YC_2^T$, $\hat{A} = A + \frac{1}{\gamma^2} B_1 B_1^T X$, and X and Y are stabilising solutions to,

$$XA + A^T X + C_1^T C_1 - X(B_2 B_2^T - \gamma^{-2} B_1 B_1^T)X = 0$$

and

$$Y\hat{A}^T + \hat{A}Y + B_1 B_1^T - Y\big(C_2^T C_2 - \gamma^{-2} F^T F\big)Y = 0,$$

satisfies $\|T_{w\to z}\|_\infty \leq \gamma$.

K Glover, 2013
F. Forni (no change), 2019