Version SSS/3

EGT3
ENGINEERING TRIPOS PART IIB

Thursday 30 April 2015 9.30 to 11

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Let $x(n)$ be a sequence of independent and identically distributed (i.i.d.) symbols where $x(n) = -1$ with probability 0.5 and $x(n) = 1$ with probability 0.5. The symbols $x(n)$ are transmitted through a finite impulse response communications channel. The output of the channel is

$$u(n) = \sum_{m=0}^{M-1} a_m x(n-m) + w(n)$$

where $w(n)$ are i.i.d. zero-mean random variables with variance σ^2.

(a) Let $\mathbf{u}(n) = [u(n), u(n-1), \ldots, u(n-L+1)]^T$. Find the Wiener filter $\mathbf{h}$ of length L that minimises the intersymbol interference $x(n) - \mathbf{h}^T \mathbf{u}(n)$. [15%]

(b) Let $a_m = 1/M$ for all m.

- (i) Find $\mathbf{p} = \mathbf{E}\{\mathbf{u}(n)x(n)\}$. [10%]
- (ii) Let $z(n) = \sum_{m=0}^{M-1} a_m x(n-m)$. Find $\mathbf{E}\{z(n)z(n+k)\}$ for $k \geq 0$. [15%]
- (iii) Find $\mathbf{R} = \mathbf{E}\{\mathbf{u}(n)\mathbf{u}(n)^T\}$. [25%]

(c) Describe, with equations, an implementation of the Least Mean Square (LMS) algorithm to learn the Wiener solution. [15%]

(d) Discuss the performance of this LMS algorithm if the coefficients a_m were not constant but time-varying. [10%]

(e) Give a bound on the LMS step-size, in terms of the parameters of this process, that ensures the LMS filter coefficients converge in expected value. [10%]

2 Consider the following two first-order autoregressive, or AR(1), processes

$$x(n) = \alpha x(n-1) + u(n)$$
$$z(n) = \beta z(n-1) + u(n)$$

where $u(n)$ are independent and identically distributed zero-mean random variables with variance σ^2.

(a) Let $y(n) = x(n) + z(n)$. Find h_0 and h_1 that minimise

$$\mathbf{E}\left\{(y(n) - h_0 u(n) - h_1 u(n-1))^2\right\}.$$

[30%]

(b) Give the recursive least squares (RLS) implementation to learn the optimal values of h_0 and h_1. [20%]

(c) Let $y(n) = x(n)z(n)$.

(i) The signal $y(n)$ can be written explicitly in terms of the random variables $u(n)$ only. Find the coefficients of $u(n)^2$ and $u(n)u(n-1)$ for this expansion of $y(n)$. [10%]

(ii) Assume $\mathbf{E}\{u(n)^4\} = \sigma^4$. Find h_0 and h_1 that minimise

$$\mathbf{E}\left\{\left(y(n) - h_0 u(n)^2 - h_1 u(n)u(n-1)\right)^2\right\}.$$

You may use the fact that $\mathbf{E}\{y(n)u(n)^2\} = \sigma^4(1-\alpha\beta)^{-1}$. [40%]

3 Consider the second-order autoregressive, or AR(2), process

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + w_n$$

where w_n are independent and identically distributed zero-mean random variables with variance σ^2.

(a) For $a_1 = 0$, find the range of values for a_2 so that $\{x_n\}$ is a wide-sense stationary (WSS) process. [10%]

(b) Find the expression for the autocorrelation function $R_{XX}[k] = \mathbf{E}\{x_n x_{n+k}\}$ for $k = 0, 1, 2$ in terms of a_1, a_2 and σ^2. [15%]

(c) The estimated autocorrelation sequence for an unknown WSS random process is

$$\hat{R}[0] = 1, \quad \hat{R}[2k] = 0.8^{|k|}$$

while $\hat{R}[k] = 0$ for odd values of k.

(i) Using these autocorrelation estimates, solve for a_1, a_2 and σ^2 for the AR(2) model. [15%]

(ii) Find the corresponding power spectral density (PSD) estimate $\hat{S}_X(e^{j\omega})$ from these values of a_1, a_2 and σ^2. [10%]

(iii) Find the expression for the PSD when estimated directly using

$$\hat{R}[-N+1], \ldots, \hat{R}[N-1]$$

for the two cases when N is finite and N is infinite. [40%]

(iv) What is the relationship between the two PSD estimates in parts (ii) and (iii) when N is finite? [10%]

4 Consider the autoregressive moving average, or ARMA(2,1), process

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + b_0 w_n + b_1 w_{n-1}$$

where w_n are independent and identically distributed zero-mean unit-variance random variables.

(a) Find $\mathbf{E}\{w_{n+k} x_n\}$ for $k = -1, 0, 1, \dots$ [20%]

(b) Find $R_{XX}[k] = \mathbf{E}\{x_{n+k} x_n\}$ for $k \geq 0$. [20%]

(c) Find a_1 and a_2 in terms of the autocorrelation of the signal x_n. [10%]

(d) Explain how a solution for b_0 and b_1 can now be found. [15%]

(e) Using the method of parts (c) and (d), find the minimum number of autocorrelation values $\mathbf{E}\{x_{n+k} x_n\}$ that are needed to solve for the parameters of an ARMA(P,Q) model. (Hint: the autocorrelation function of the ARMA(P,Q) model is

$$R_{XX}[r] = \sum_{p=1}^{P} a_p R_{XX}[r-p] + \sum_{q=0}^{Q} b_q h_{q-r}$$

where $h_0, h_1, \dots$ is the impulse response of the ARMA(P,Q) model.) [35%]

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