EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 20 April 2016 2 to 3.30

**Module 4F7**

**DIGITAL FILTERS AND SPECTRUM ESTIMATION**

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

**STATIONERY REQUIREMENTS**
Single-sided script paper

**SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM**
CUED approved calculator allowed
Engineering Data Book

**10 minutes reading time is allowed for this paper.**

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.**

1 (a) Describe briefly the principles behind the *nonparametric* power spectral estimation method. Your discussion should include the correlogram, periodogram and possible improvement strategies. [30%]

(b) It is proposed to estimate the power spectrum of a wide-sense stationary random process by first multiplying the data $x_n$ with a window function $w_n$ having length $N$, i.e. $w_n = 0$ for $n < 0$ and $n > N-1$, so that

$$x_n^w = w_n x_n \,.$$

The autocorrelation function is then estimated as

$$\hat{R}_{XX}[|k|] = \begin{cases} \frac{1}{N}\sum_{n=0}^{N-1} x_n^w x_{n+|k|}^w, & k = -N+1, \ldots, -1, 0, 1, \ldots, N-1\,, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Show that the expected value of the autocorrelation function estimate is given by

$$E[\hat{R}_{XX}[|k|]] = R_{XX}[k]\frac{1}{N}\sum_{n=0}^{N-1-|k|} w_n w_{n+|k|}$$

where $R_{XX}[k]$ is the true autocorrelation function for the process, and hence explain whether this estimator is biased or not. [30%]

(ii) The power spectrum estimate $\hat{S}_X(\mathrm{e}^{j\theta})$ is obtained by taking the DTFT of the estimated autocorrelation function $\hat{R}_{XX}[k]$.
Show that the expected value of the corresponding power spectrum estimate is:

$$E[\hat{S}_X(\mathrm{e}^{j\theta})] = \frac{1}{2\pi N} S_X(\mathrm{e}^{j\theta}) * |W(\mathrm{e}^{j\theta})|^2$$

where $S_X(\mathrm{e}^{j\theta})$ is the true power spectrum of the random process, $W(\mathrm{e}^{j\theta})$ is the DTFT of the window function $w_n$, and $*$ denotes the convolution operator. [25%]

(iii) Explain the advantages and disadvantages of this method for power spectral estimation in comparison with the standard periodogram estimator. [15%]

2 (a) Describe the autoregressive moving average (ARMA) class of signal model, explaining how to obtain the power spectrum of an ARMA process and any advantages of such an approach compared to nonparametric approaches. [30%]

(b) An ARMA($P$,$Q$) model has the following digital filtering equation:

$$x_n = -\sum_{p=1}^{P} a_p x_{n-p} + \sum_{q=0}^{Q} b_q w_{n-q}\,,$$

where $\{w_n\}$ is zero mean white noise with unity variance, and the filter is assumed stable.

(i) Explain carefully why it is not necessary to include a variance parameter for the white noise process $\{w_n\}$ in the above ARMA formulation. [10%]

(ii) Show that the ARMA model autocorrelation function obeys the following difference equation:

$$R_{XX}[r] + \sum_{p=1}^{P} a_p R_{XX}[r-p] = \sum_{q=0}^{Q} b_q h_{q-r}\,,$$

where $h_r$ is a particular function of the ARMA filter that should be carefully defined. Explain why the term $\sum_{q=0}^{Q} b_q h_{q-r}$ must always be zero for $r > Q$. [30%]

(c) An ARMA(1,1) model is to be estimated from autocorrelation data.

(i) Express the first two terms $h_0$ and $h_1$ from the ARMA(1,1) model in terms of the coefficients $\{a_p\}$ and $\{b_q\}$. [10%]

(ii) Some values of the autocorrelation function for an ARMA(1,1) process are given by

$$R_{XX}[0] = 1,\ R_{XX}[1] = -0.4,\ R_{XX}[2] = 0.2,\ R_{XX}[3] = -0.1\,.$$

Use the result of part (b)(ii) and your expressions for $h_0$ and $h_1$ to determine the coefficients of the corresponding ARMA(1,1) model. You are given that $b_0$ equals 2. [20%]

3 Consider the following recursive algorithm:

$$\mathbf{h}(n) = \mathbf{h}(n-1) + \mu\tilde{\mathbf{R}}^{-1}(\mathbf{p} - \mathbf{R}\mathbf{h}(n-1)),$$

where $\mathbf{R}$ and $\mathbf{p}$ are a postive definite matrix (input correlation matrix) and a vector (crosscorrelation vector between input signal and reference) of appropriate dimensions, respectively. The positive definite matrix $\tilde{\mathbf{R}}$ is assumed to be an approximation or estimate of the true correlation matrix $\mathbf{R}$. Moreover, it is assumed that $\tilde{\mathbf{R}}$ can be expressed as $\tilde{\mathbf{R}} = \mathbf{Q}\tilde{\Lambda}\mathbf{Q}^T$, where the matrix $\mathbf{Q}$ is an orthonormal matrix and contains the eigenvectors of the original correlation matrix $\mathbf{R}$, and $\tilde{\Lambda}$ is approximated as a diagonal matrix.

(a) Assuming the coefficient vector $\mathbf{h}(n)$ of the algorithm converges towards a limit $\mathbf{h}_{\text{opt}}$, find an expression for $\mathbf{h}_{\text{opt}}$. [10%]

(b) (i) Based on the eigenvalue decomposition (modal decomposition) of $\mathbf{R}$ and the above expression decomposition of $\tilde{\mathbf{R}}$, obtain a recursion for the misalignment $\mathbf{h}(n) - \mathbf{h}_{\text{opt}}$ in the corresponding eigendomain, and find the limits for the choice of the stepsize $\mu$ ensuring convergence of the algorithm whatever initial vector $\mathbf{h}(0)$ is chosen. [30%]

(ii) Discuss the extreme cases $\tilde{\mathbf{R}} = \mathbf{R}$ and $\tilde{\mathbf{R}} = \mathbf{I}$. Distinguish in this discussion between the use of a common stepsize for all modes, and modal stepsizes in which different step sizes may be chosen for each mode. [15%]

(iii) In the case $\tilde{\mathbf{R}} = \mathbf{I}$ and a single stepsize for all modes, express the range of stepsize in terms of a signal variance rather than eigenvalues. [15%]

(c) In practical applications, the quantities $\mathbf{R}$ and $\mathbf{p}$ are typically not known in advance. Moreover, they can be time-varying.

(i) How can the above recursive algorithm be approximated to obtain practical algorithms such as the Least-Mean-Square (LMS) algorithm? State the relation explicitly using equations. [10%]

(ii) How is the matrix $\tilde{\mathbf{R}}$ defined for the LMS algorithm? [10%]

(iii) How should the matrix $\tilde{\mathbf{R}}$ be defined in order to obtain a Recursive-Least-Square (RLS)-like algorithm? Note that in this case, it will be required to handle nonstationary environments. [10%]

4 (a) In the standard adaptive filtering problem we have an input signal $\{u(n)\}_{n=0}^{\infty}$, a reference signal $\{d(n)\}_{n=0}^{\infty}$, and a Finite Impulse Response (FIR) filter $\{h_m\}_{m=0}^{M-1}$ of length $M$. Describe the setup of the general adaptive filter, including the error criterion/cost function in vector notation, and illustrate it by a simple block diagram. Also explain briefly the main conceptual difference between a Wiener filter and an adaptive filter implementation. [15%]

(b) Name the four basic classes of application within the framework of part (a) and describe any three of them with the aid of block diagrams. Give one practical example for each class of applications. [15%]

(c) One of the most popular adaptive filtering algorithms is the Least-Mean-Square (LMS) algorithm. Explain the main ideas behind the LMS algorithm and give the coefficient update equation. How is it obtained from the cost function in part (a)? (No detailed derivation is required.) [10%]

(d) The Normalised LMS (NLMS) algorithm is closely related to the LMS algorithm.

(i) Give the coefficient update equation of the NLMS algorithm. What is the advantage of the NLMS algorithm over the LMS? [15%]

(ii) Describe how the NLMS coefficient update equation can be interpreted as a projection mechanism. Give a geometrical illustration of this projection. [15%]

(e) The idea of interpreting the NLMS coefficient update as a projection operation (as in part (d)(ii)) can be generalised to yield a whole class of improved adaptation algorithms.

(i) Explain how the NLMS projection idea can be generalised to improve performance. Give a graphical illustration of the projections involved. [15%]

(ii) Give the coefficient update equation for the resulting algorithm. [15%]

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Answers

1

2 (b)(i)

$$h_0 = b_0, h_1 = -a_1 h_1 = -a_1 b_0 + b_1$$

(ii)

$$a_0 = 0.5, b_1 = 0.05$$

3

4