Version SSS/4

EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 26 April 2017   2 to 3.30

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 A source transmits a sequence of wide sense stationary random symbols $x(n)$ where $\Pr\{x(n)=1\} = \Pr\{x(n)=-1\} = 0.5$ and $E\{x(n)x(n+k)\} = \beta^{|k|}$ where $0 < \beta < 1$.

$x(n)$ is transmitted through a communications channel and the output of the channel is

$$d(n) = \sum_{m=0}^{\infty} a_m x(n-m) + w(n)$$

where $w(n)$ is a sequence of independent and identically distributed (i.i.d.) zero-mean random variables with variance σ^2.

A local copy of the transmitted symbols are available at the destination which is to be used to estimate the channel impulse response.

(a) Let $\mathbf{x}(n) = [x(n),\ x(n-1),\ \ldots,\ x(n-M+1)]^T$. Find $\mathbf{p} = E\{\mathbf{x}(n)d(n)\}$ and $\mathbf{R} = E\{\mathbf{x}(n)\mathbf{x}(n)^T\}$. [15%]

(b) Show that the Wiener filter that minimises the error of the channel approximation $d(n) - \mathbf{h}^T\mathbf{x}(n)$ satisfies the equation $\mathbf{Rh} = \mathbf{p}$. (The filter $\mathbf{h}$ has length M.) [15%]

(c) For $M = 1$ solve for the Wiener filter and explain why the Wiener solution is not a_0. [10%]

(d) Describe, with equations, an implementation of the Least Mean Square (LMS) algorithm to learn the Wiener solution. Discuss the performance of this LMS algorithm if the coefficients a_m were not constant but time-varying. [15%]

(e) Using $\mathbf{Rh} = \mathbf{p}$ or otherwise, show that the Wiener solution is the channel impulse response in the limit as M tends to infinity. [25%]

(f) Assume now that the source symbols are i.i.d.

(i) Find the Wiener filter for this new case. [10%]

(ii) Find the minimum mean square error and sketch it as a function of M. [10%]

2 Let θ_0 be a random variable with finite variance and mean. Let $\{w(n)\}$ and $\{v(n)\}$ be two sequences of independent and identically distributed (i.i.d.) random variables with $E\{w(n)\} = E\{v(n)\} = 0$ and $E\{w(n)^2\} = \sigma_w^2$, $E\{v(n)^2\} = \sigma_v^2$.

(a) Let $y_0 = \theta_0 + v(0)$ be a noisy observation of θ_0. Find the constant c that minimises the mean square error (MSE) $E\left\{(\hat{\theta}_0 - \theta_0)^2\right\}$ of the estimate of θ_0 given by $\hat{\theta}_0 = cy_0$. Give the minimum MSE. [15%]

(b) Let $\theta_1 = \theta_0 + w(1)$ and consider the estimate of θ_1 given by $\tilde{\theta}_1 = d\hat{\theta}_0$ where $\hat{\theta}_0$ is your minimum MSE estimate from part (a). Find the constant d that minimises the MSE $E\left\{(\tilde{\theta}_1 - \theta_1)^2\right\}$ and give the MSE of $\tilde{\theta}_1$. [10%]

(c) Let $y_1 = \theta_1 + v(1)$. Consider the following *updated* estimate of θ_1 that uses observation y_1

$$\hat{\theta}_1 = K\tilde{\theta}_1 + Ly_1$$

where $\tilde{\theta}_1$ is your minimum MSE estimate from part (b).

(i) Find the relationship between K and L so that the estimate $\hat{\theta}_1$ is unbiased.

(ii) Find K and L that minimises the MSE $E\left\{(\hat{\theta}_1 - \theta_1)^2\right\}$.

[50%]

(d) Consider the model

$$y_n = b_n\theta_n + v(n)$$
$$\theta_{n+1} = a_{n+1}\theta_n + w(n+1)$$

for $n \geq 0$ where a_n and b_n are known constants. Explain how your results from parts (a) to (c) can be used to obtain the sequence of estimates $\hat{\theta}_n$ of θ_n. [25%]

3 Consider the AR(P) process

$$x_n = \sum_{i=1}^{P} a_i x_{n-i} + w_n$$

where w_n is a white noise sequence with variance σ^2.

(a) Let $p(x_0, \ldots, x_{P-1})$ be the probability density function of P variables of the AR(P) process.

(i) Define *strictly stationary*.

(ii) Assuming the AR(P) process is strictly stationary, describe, with equations, a procedure to find $p(x_0, \ldots, x_{P-1})$. Justify any further assumptions made on the noise process w_n.

[25%]

(b) Consider a strictly stationary AR process with $P = 2$ and $a_1 = 0$.

(i) Find $p(x_0, x_1)$. [15%]

(ii) Let $p(x_2, \ldots, x_n | x_0, x_1)$ denote the conditional probability density function of of $x_2, \ldots, x_n$ given the values of x_0 and x_1. Find the maximum likelihood estimate (MLE) of a_2 and of σ^2. [40%]

(iii) Relate the MLE of a_2 and of σ^2 to the estimates given by the Yule-Walker method. [20%]

4 (a) Show that the variance of the periodogram estimate is approximately equal to the square of the power spectrum for a Gaussian white noise process, i.e. show that

$$\mathrm{var}\left(\hat{S}_X(e^{j\omega})\right) \approx S_X(e^{j\omega})^2$$

where $\hat{S}_X(e^{j\omega})$ is the periodogram estimate and $S_X(e^{j\omega})$ is the power spectrum of white Gaussian noise. [35%]

(b) The estimate of the power spectrum of a random process $\{x_n\}$ is being updated sequentially as follows:

$$\hat{S}^{(k)}(e^{j\omega}) = (1-\gamma_k)\hat{S}^{(k-1)}(e^{j\omega}) + \gamma_k \frac{1}{N}\left|\sum_{n=0}^{N-1} x_n^{(k)} e^{-jn\omega}\right|^2$$

where $x_n^{(k)} = x_{n+N(k-1)}$ is the nth sample of the kth block of N data points. This equation is initialised at $k=1$ with $\hat{S}^{(0)}(e^{j\omega}) = 0$.
Assume $\{x_n\}$ is white Gaussian noise.

(i) Find the the variance of $\hat{S}^{(k)}(e^{j\omega})$ when $\gamma_k = 1/k$ and also when $\gamma_k = a$ where $0 < a < 1$. [30%]

(ii) Find the limit of these variances as k tends to infinity. [10%]

(c) Assume that $x_n = w_n + w_{n-1}$ where $\{w_n\}$ is white noise. Find the expected value of the periodogram estimate $\frac{1}{N}\left|\sum_{n=0}^{N-1} x_n e^{-jn\omega}\right|^2$ for frequency $\omega = 0$. [25%]

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