Version SSS/4

EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 24 April 2019 2 to 3.40

Module 4F7

STATISTICAL SIGNAL ANALYSIS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Let X be a zero mean random variable. Let $Y_1, \ldots, Y_n$ be a further collection of n random variables and let $\mathbf{E}\{Y_i\} = m_i$. A linear estimate of X using $Y_1, \ldots, Y_n$ is

$$\hat{X}_n = h_1\,(Y_1 - m_1) + \ldots + h_n\,(Y_n - m_n).$$

(a) The constants $h_1, \ldots, h_n$ that minimise the error $\mathbf{E}\left\{(X - \hat{X}_n)^2\right\}$ can be expressed as

$$A \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} = b.$$

Find the matrix A and vector b. [30%]

(b) Let Y_n satisfy

$$\mathbf{E}\{Y_i Y_n\} = \mathbf{E}\{Y_i\}\mathbf{E}\{Y_n\}, \qquad \text{for } i = 1, \ldots, n-1.$$

(i) Show that the minimising h_n satisfies

$$h_n = \left(\mathbf{E}\{Y_n Y_n\} - \mathbf{E}\{Y_n\}^2\right)^{-1} \mathbf{E}\{XY_n\}.$$

[10%]

(ii) Let $\hat{X}_{n-1}$ be the best linear estimate of X using $Y_1, \ldots, Y_{n-1}$. Find the equation that updates $\hat{X}_{n-1}$ to $\hat{X}_n$. [20%]

(c) Let X be a zero mean Gaussian random variable. A sensor provides noisy measurements of X,

$$Y_i = \text{sign}(X) + W_i$$

for $i = 1, \ldots, n-1$ where W_i are independent zero mean random variables with variance σ^2 and $\text{sign}(X) = 1$ if $X \geq 0$ and $\text{sign}(X) = -1$ otherwise.

(i) Find $\hat{X}_{n-1}$. [20%]

(ii) Measurement Y_n is generated by a different sensor, $Y_n = |X| + W_n$. Find $\hat{X}_n$. [20%]

2 Consider the following hidden Markov model (HMM) where X_n is a two-state hidden Markov chain, $X_n \in \{-1, 1\}$, with transition probability matrix

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{bmatrix}.$$

Let Y_n be the observed process, $Y_n \in \{-1, 1\}$, where

$$p(y_n|x_n) = \begin{cases} 1-\beta & \text{if } y_n = x_n, \\ \beta & \text{if } y_n \neq x_n. \end{cases}$$

Assume $X_0 = x_0$.

(a) Given $p(x_n|y_1, \ldots, y_{n-1})$, find $p(x_{n+1}|y_1, \ldots, y_n)$. [20%]

(b) Given $p(y_{n+1}, \ldots, y_T|x_{n+1})$, find $p(y_n, \ldots, y_T|x_n)$. [20%]

(c) Find $p(x_n|y_1, \ldots, y_T)$. [10%]

(d) Given a sequence of values $x_1, y_1, \ldots, x_T, y_T$, find the value of β that maximises $p(x_1, y_1, \ldots, x_T, y_T|x_0)$. [20%]

(e) Assume α is known. Give the Expectation-Maximisation algorithm for finding the value of β that maximises $p(y_1, \ldots, y_T|x_0)$. [20%]

(f) Let S_n be the price of a financial asset (e.g. a share) and let

$$Y_n = \begin{cases} 1 & \text{if } S_n \geq S_{n-1}, \\ -1 & \text{otherwise.} \end{cases}$$

Describe what X_n signifies in this application and the impact, on de-noising the data, of choosing different values of β from the range $0 \leq \beta \leq 1$. [10%]

3 Consider the following state-space model with Gaussian noise: for $k \geq 0$,

$$X_{k+1} = X_k + W_{k+1},$$
$$Y_k = X_k + V_k,$$

where $\{W_k\}$ is an independent and identically distributed (i.i.d.) sequence of Gaussian random variables with mean zero and variance σ_w, $\{V_k\}$ is an i.i.d. sequence of Gaussian random variables with mean zero and variance σ_v. X_0 is an independent Gaussian random variable with mean μ_0 and variance σ_0.

(a) Assume that the conditional probability density function (pdf) $p(x_n|y_0, \ldots, y_n)$ is a Gaussian pdf with mean μ_n and variance σ_n.

(i) Find $p(x_{n+1}|y_0, \ldots, y_n)$. [20%]

(ii) Find $p(x_{n+1}|y_0, \ldots, y_{n+1})$ and give its mean μ_{n+1} and variance σ_{n+1}. [20%]

Hint: You may use the following facts about two independent random variables U_1 and U_2 with pdfs $p_i(u_i)$. If $U_1 + U_2 = y$ then $p(y) = \int p_2(y - u_1)p_1(u_1)\mathrm{d}u_1$.
If U_1 is $\mathcal{N}(\mu_1, \sigma_1)$ and U_2 is $\mathcal{N}(0, \sigma_2)$, where $\mathcal{N}(\mu_1, \sigma_1)$ denotes a Gaussian random variable with mean μ_1 and variance σ_1, then the conditional pdf $p_2(y - u_1)p_1(u_1)/p(y)$ is Gaussian with mean $(\sigma_1 y + \sigma_2\mu_1)/(\sigma_1 + \sigma_2)$ and variance $(\sigma_1\sigma_2)/(\sigma_1 + \sigma_2)$.

(b) Show that if $\sigma_{n+1} = \sigma_n$ then $\sigma_{n+1}^2 \leq \sigma_w\sigma_v$. How might the bound $\sigma_w\sigma_v$ be useful in practice? [30%]

(c) Show that μ_n is a linear combination of $y_0, \ldots, y_n$ and μ_0. [10%]

(d) Show that μ_n is the best linear estimate of X_n of the form

$$\hat{x}_n = h\ \mu_0 + h_0\ y_0 + \ldots + h_n\ y_n$$

that minimises the error $\int(\hat{x}_n - x_n)^2 p(x_n|y_0, \ldots, y_n)\mathrm{d}x_n$. [20%]

4 Let $X_0, X_1, \ldots$ be a Markov chain with values in $\{0, 1, 2, \ldots\}$ with the following transition probabilities

$$\Pr(X_{n+1} = j | X_n = i) = \begin{cases} \alpha & \text{if } j = i + 1, \\ 1 - \alpha & \text{if } j = i - 1, \end{cases}$$

when $X_n = i > 0$. For $X_n = 0, \Pr(X_{n+1} = 1 | X_n = 0) = \alpha$ and $\Pr(X_{n+1} = 0 | X_n = 0) = 1-\alpha$. Assume $\Pr(X_0 = k) = \exp(-\lambda)\lambda^k / k!$, i.e. X_0 has the probability mass function (pmf) of a Poisson random variable.

Let

$$Y_k = X_k + V_k$$

for $k \geq 0$ where $V_0, V_1, \ldots$ are independent zero mean Gaussian random variables with variance σ.

(a) Given the pmf $p(x_n | y_0, \ldots, y_n)$, find $p(x_{n+1} | y_0, \ldots, y_{n+1})$. [20%]

(b) Let $X_{0:n}^1, \ldots, X_{0:n}^N$ be N independent samples from $p(x_0, \ldots, x_n)$.

(i) Find an unbiased estimate $\hat{p}(y_0, \ldots, y_n)$ of $p(y_0, \ldots, y_n)$. [10%]

(ii) Give the importance sampling (IS) estimate of $p(x_0, \ldots, x_n | y_0, \ldots, y_n)$. [10%]

(iii) Give the IS estimate of the value α such that

$$\sum_{x_0=0}^{\infty} \cdots \sum_{x_n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}\alpha} \log p(x_1, \ldots, x_n | x_0) p(x_0, \ldots, x_n | y_0, \ldots, y_n) = 0.$$

[30%]

(iv) Using sequential importance sampling with re-sampling, extend the IS estimate in part (b)(ii) to an IS estimate of $p(x_0, \ldots, x_{n+1} | y_0, \ldots, y_n)$. [10%]

(v) Give the IS estimate $\hat{p}(y_{n+1} | y_0, \ldots, y_n)$ of the conditional probability density function $p(y_{n+1} | y_0, \ldots, y_n)$. Show that the product of estimates

$$\hat{p}(y_{n+1} | y_0, \ldots, y_n) \hat{p}(y_0, \ldots, y_n)$$

is an unbiased estimate of $p(y_0, \ldots, y_{n+1})$. [20%]

END OF PAPER

THIS PAGE IS BLANK